Abstract

Oscillatory motion is exhibited in physical systems in a vast range of scales all-encompassing from molecular vibrations to celestial mechanics. Propagating wave phenomena are also encountered in many systems embracing man-made linear and non-linear devices used as information carriers, as well as a myriad of acoustical vibrations and electromagnetic phenomena observed in nature. The dynamic equilibrium between two forms of energy is shared by all these systems. Amplitude and phase categories provide a suitable conceptual and mathematical framework to describe these physical processes. Invariants arising from this formalism play a crucial role in the obtention of solutions and conserved quantities.

The amplitude and phase representation of coordinate or disturbance variables in second order differential equations leads to an Ermakov pair of equations. Under appropriate circumstances these equations may be decoupled into a second order non-linear differential equation for the amplitude and a third order nonlinear equation for the phase. Various results scattered in the literature will be presented here in a unified manner as well as several new developments in three broad areas, namely classical dynamical systems, quantum mechanics and electromagnetic propagation.

Classical dynamical systems: The one dimensional time dependent harmonic oscillator is studied in detail. The Ermakov-Lewis invariant or a closely related orthogonal functions invariant is economically derived. The nonlinear superposition principle is revised. General solutions, given a particular solution, to the non linear amplitude and phase equations are presented. The analysis of these results lead to novel analytical solutions of the time dependent oscillator equation with a sub-period monotonic parameter. This procedure, as we shall demonstrate, seems promising for the development of analytical solutions with more general parameters. The indeterminacy of the amplitude and phase functions has been recently addressed and is further discussed in this paper. The relationship between exact invariants and previously known adiabatic invariants is evaluated.

Quantum domain: The orthogonal functions invariant is generalized to the quantum realm. The time dependent quantized quadratic Hamiltonian gives rise to two
linear invariants that may be casted into a quadratic form. The time dependent Schrodinger equation with one spatial dimension is solved with this formalism. It is shown that in the time dependent case, the Ermakov-Lewis invariant plays the role that the Hamiltonian does in the time independent case. The invariant operator is expressed in terms of amplitude and phase operators. Squeezed and coherent states are simply dealt with unitary transformations that stem from the invariant form. The evolution of minimum uncertainty states are studied under an abrupt frequency or mass change.

Wave propagation: The time dependent oscillator problem is shown to be formally equivalent to the electromagnetic propagation of monochromatic waves in one dimension. The nonlinear amplitude differential equation allows for an arbitrary refractive index variation. A hyperbolic tangent refractive index profile will be treated in detail. The orthogonal functions invariant translates into a conserved quantity for the 1+3 dimensional scalar wave equation. The assessed quantity and its associated flow can then be evaluated using different criteria. The important topic of phase retrieval from intensity measurements and the irradiance propagation equation is discussed. This formalism leads to radiometric quantities that differ for ultra-short pulses from the prevailing optical radiometric theory.

1 Introduction

The statement of a physical problem in terms of adequate variables is most relevant in order to i) convey a conceptual meaning to the variables that are involved; ii) describe and if possible, predict the evolution of the system; iii) establish measurable parameters; iv) impose boundary conditions, either given a priori or specific for a given purpose; v) find analytical, numerical or approximate solutions. These issues constitute our ultimate understanding of physical phenomena. Clearly or unambiguously defined quantities in diverse circumstances establish powerful categories such as energy, neighborhood or velocity. A well defined behavior of these variables under general considerations prove them extremely useful. Such behavior may be the fulfillment of an inequality or invariance under general transformations; Think for example in entropy increase, energy constancy or metric invariance. In this essay, the position or disturbance will be written in terms of amplitude and phase variables. The relevance of these quantities and their adequacy will be discussed under the before mentioned premises. The usefulness of the amplitude approach in several specific problems will be presented.

The amplitude and phase representation of periodic or quasi periodic phenomena is concomitant to these type of systems. Two general scopes are encompassed. i) Non propagating problems such as mechanical vibrations (molecule vibration, harmonic oscillators) or trajectories in discrete mechanical systems encountered in celestial mechanics [1] and terrestrial based devices. ii) Propagating problems that involve diverse wave phenomena from acoustics [2] and continuum mechanics to electromagnetic propagation [3]. It is often highlighted that this representation allows for the separation of a rapidly oscillating term in the phase and a slowly varying amplitude term. This approximation is coined as the slowly varying envelope in optics and transmission theory or the WKB method in quantum
mechanics.

Periodic or quasi periodic phenomena may be viewed in terms of physical systems that exchange two forms of energy. The *dynamic equilibrium* between two forms of energy, involves the back and forth transformation of one form of energy into another as time evolves. This process is sometimes referred to as *imbalance* in the literature [4]. It corresponds for example, to kinetic and potential energy in classical mechanics, electric and magnetic fields in electromagnetism or the real and imaginary parts of the wave function in quantum physics. In this perspective, heuristically speaking, the phase is related to the energy transfer between two kinds of manifestation whereas the amplitude is associated with the overall energy content in both fields.

Conserved quantities, in particular, conservation of matter is an early concept in physical thought, already recorded by Lucretius circa 55 BC [5, p.35]. However, the closely related ideas of energy and momentum conservation appeared late and timidly in physical thought. Constants of motion were first established by Huygens and Wallis regarding elastic collisions in 1668 [6, ch.17, p.219]. The concept, named *vis viva* after Leibniz, was understood as a form of energy, namely, kinetic energy by Thomson and Tait as late as 1867 [7, p.35]. Transformation between different kinds of energy such as the mechanical equivalence of heat where put forward by Mayer in 1842 [6, p.239] and thereafter developed by Joule and the Scottish school [8]. The conservation of energy and momentum paved the way to the notion of invariants in physics and mathematics during the twentieth century. Adiabatic invariants, initially developed in statistical mechanics to deal with an ensemble of Planck oscillators, are approximately constant for slowly varying time dependent parameters. Exact invariance of diverse quantities such as the metric or the charge under certain transformations, have become a central formulation in present day theories. From a mathematical point of view, invariants of second order differential equations were initially addressed by Ermakov [9]. Some succinct results were later presented by Milne [10] and Pinney [11]. The study of this field flourished after the publications of Lewis [12], Gunther and Leach[13], Ray and Reid [14] and other workers. This approach was linked to symmetry groups by Noether’s theorem [15, 16] and Lie algebras [17]. The amplitude and phase representation, as we shall see, is concomitant to Ermakov systems and the theory of invariants. The Ermakov-Pinney equation also appears in diverse scenarios such as scalar field cosmologies [18] or accelerator physics [19].

Despite the before mentioned merits, the amplitude and phase representation has an important disadvantage. The coordinate or disturbance linear differential equation is not always separable into two differential equations for the amplitude and the phase. Furthermore, even if they can be decoupled, non linear differential equations in these variables are obtained [20]. These drawbacks have been partially overcome in recent decades due to the development of appropriate techniques towards the solution of nonlinear differential equations.

These general ideas will be discussed here in a twofold restricted context. On the one hand, time dependent discrete phenomena will be limited to one dimension. Nonetheless, it should be mentioned that important efforts in higher dimensions have been made
In propagating continuum phenomena, monochromaticity is imposed and one spatial dimension considered in detail. The amplitude and phase representation of the scalar wave equation is discussed in 3+1 dimensions. On the other hand, the potential will be restricted to a linear form in the coordinate variable, albeit with an arbitrary real functional dependence with respect to the variable of integration. Other potentials considered within the Ermakov formalism will only be briefly discussed.

The time dependent classical harmonic oscillator in one dimension is studied in the second section. The orthogonal functions invariant and related conserved quantities are derived for a generalized Ermakov system, a dissipative system and a problem with time dependent mass. The nonlinear superposition principle is revised in this context and novel approximate analytical solutions are presented. These solutions may be readily applied, for example, to conformational transitions in molecules [24]. The measurement of the classical amplitude and phase variables are shown not to be uniquely determined in the propagation-less case.

The quantum mechanical time dependent harmonic oscillator is addressed in the third section. Invariant quantities are reformulated with the appropriate operators. The Ermakov Lewis invariant is shown to play the role that the Hamiltonian does in the time independent case. Namely, that the time evolution of the operators is proportional to the commutation brackets of the operator with the invariant. Minimum uncertainty states are evolved in time subject to a rapid frequency or mass variation.

The fourth section undertakes propagating phenomena. The simplest 1+1 (time plus one spatial coordinate) dimensional wave propagation in a deterministic inhomogeneous medium is first approached. In subsequent subsections, we deal with the 1+3 scalar wave equation and ultimately with the vector nonlinear amplitude and phase equations. Conserved quantities and their associated flow are evaluated under different schemes. The obtention of the wavefront phase from non-interferometric amplitude (intensity) measurements is treated in the paraxial approximation. The energy content of finite wave-trains is assessed with the present complementary fields formalism and compared with the WMW radiometric theory. It is finally suggested that experiments with ultra short laser pulses should elucidate the differences between the two approaches.

2 Classical dynamical systems

2.1 Time dependent harmonic oscillator

The so called Ermakov-Lewis invariant or a closely related orthogonal functions invariant have been derived in a variety of ways. It has been obtained using Kruskal theory, symplectic group transformations, Noether’s theorem [25] and the orthogonal functions technique. The former procedure proves that a previously known adiabatic invariant is an exact invariant [12]. The second method invokes a time-dependent canonical transformation, which makes the Hamiltonian constant [26]. In the third approach, Noether’s theorem relates the conserved quantities of the Lagrangian system to the symmetry groups that leave the ac-
tion invariant [15]. In the latter formulation, that we recreate here below, the Wronskian is constructed from linearly independent solutions and rewritten in terms of the appropriate variables.

2.2 Invariant derivation

Harmonic phenomena with variable parameters are encountered in many fields of physics. Oscillators with time dependent potentials and propagation of monochromatic waves in deterministic inhomogeneous media are two such examples. The non autonomous second order differential equation governing these phenomena in one dimension is

$$\frac{d^2q}{dt^2} + \Omega^2 q = 0,$$  \hspace{1cm} (1)

where the position or disturbance is represented by $q$, the variable of differentiation may be a spatial or temporal coordinate depending on the nature of the problem and the real parameter $\Omega$ is dependent on this variable. The Wronskian of the TDHO equation (1) is

$$W \equiv q_1 \dot{q}_2 - q_2 \dot{q}_1,$$ \hspace{1cm} (2)

where the real functions $q_1, q_2$ are linearly independent solutions if $W \neq 0$ and orthogonal if they fulfill the Sturm Liouville orthogonal functions integral. Define the time dependent function $\rho$ in terms of two linearly independent solutions of the TDHO equation as

$$\rho^2 = q_1^2 + q_2^2.$$ \hspace{1cm} (3)

The Wronskian in terms of $q_1 = q$ and $\rho$ is obtained by evaluating the function $q_2$ and its derivative from the above equation

$$W = \frac{\rho}{\sqrt{\rho^2 - q^2}} (q \dot{\rho} - \rho \dot{q}),$$ \hspace{1cm} (4)

which may be rearranged as

$$2I = W^2 = W^2 \frac{q^2}{\rho^2} + (q \dot{\rho} - \rho \dot{q})^2,$$ \hspace{1cm} (5)

where $I$ is the Ermakov-Lewis invariant for the TDHO equation. This invariant $I$ is therefore equal to one half the Wronskian squared. From the derivative of the above equation an ODE for the function $\rho$ is obtained by invoking the invariant relationship (4) and the oscillator equation (1),

$$W^2 - \rho^3 (\dot{\rho} + \rho \Omega^2) = 0.$$ \hspace{1cm} (6)

This auxiliary equation together with the TDHO equation form an Ermakov pair. Equation (4) may be readily integrated to yield

$$q = \rho \cos \left( \int \frac{W}{\rho^2} dt \right).$$
The function $\gamma$ is defined as the argument of the cosine function
\[ \gamma = \int \frac{W}{\rho^2} dt, \]
the product $q = \rho \cos \gamma$ recreates the coordinate variable. The variables $\rho$ and $\gamma$ are physically interpreted as amplitude and phase quantities. The amplitude and phase representation is thus concomitant to Ermakov systems with exact invariants. The Wronskian is then given by
\[ W = \rho^2 \gamma. \]  
(7)
The frequency defined as the derivative of the phase function $\dot{\gamma} \equiv \omega$ fulfills the nonlinear differential equation
\[ \omega \ddot{\omega} - \frac{3}{2} \omega^2 + 2 \left[ \omega^2 - \Omega^2 \right] \omega^2 = 0. \]  
(8)
This last result, obtained from (6) and (7), is the starting point of the phase integral approximation [27] or phase-amplitude method [28].

The exact invariant can be extended to more general systems in various ways. On the one hand, an arbitrary driving term can be included with the restriction that the argument should be the quotient of the variables in the Ermakov pair (section 2.2.1). On the other hand, it is possible to permit absorption (section 2.2.2) or a time dependent mass (section 2.2.3) in the harmonic oscillator differential equation.

### 2.2.1 Generalized Ermakov system

The TDHO equation may be generalized to the Ermakov pair
\[ \frac{d^2 q}{dt^2} + \Omega^2 q = \frac{1}{q^3} f \left( \frac{q}{\rho} \right), \]
and
\[ \frac{d^2 \rho}{dt^2} + \Omega^2 \rho = \frac{1}{\rho^3} g \left( \frac{\rho}{q} \right), \]
where the functions $f$ and $g$ are arbitrary functions of their arguments. If $\Omega^2$ is eliminated from these two equations and the resulting equation multiplied by $q \rho \left( \dot{q} \dot{\rho} - \rho \dot{q} \right)$, then
\[
0 = \frac{1}{2} \frac{d}{dt} (q \dot{\rho} - \rho \dot{q})^2 + du q^2 \left[ \frac{\rho}{q^3} f \left( \frac{q}{\rho} \right) - \frac{q}{\rho^3} g \left( \frac{\rho}{q} \right) \right]
\]
where the variable
\[ u = \frac{\rho}{q}, \quad \frac{du}{dt} = \frac{1}{q^2} (q \dot{\rho} - \rho \dot{q}) \]
is introduced. Integration of this equation gives the generalized Ermakov invariant
\[ 2I = W_g^2 = (q \dot{\rho} - \rho \dot{q})^2 + 2 \int_{q}^{\rho} \left[ uf(u) - u^{-3} g(u) \right] du. \]  
(9)
It has been pointed out that $\Omega^2$ has to be continuous and differentiable [29]. However, it has been recently shown that it is sufficient if this function is continuous and piecewise integrable [30].

### 2.2.2 Oscillator with absorption

Absorption can be included in the TDHO equation (1) via a first order derivative term,

$$\frac{d^2q}{dt^2} + \alpha \frac{dq}{dt} + \Omega^2 q = 0.$$  \hspace{1cm} (10)

The first order derivative may be eliminated with the variable transformation $q = \kappa \exp\left[-\frac{1}{2} \int \alpha dt \right]$. If $\alpha$ is rewritten as $\alpha = \frac{d \ln \mu}{dt} = \frac{1}{\mu} \frac{d\mu}{dt}$, $\mu = \mu_{init} \exp\left(\int \alpha dt \right)$, the transformation takes the simpler form $q = \kappa / \sqrt{\mu}$. The TDHO equation $\frac{d^2 \kappa}{dt^2} + \Omega^2 \kappa = 0$ then has a time dependent parameter

$$\Omega^2 \kappa = \Omega^2 - \frac{1}{2} \frac{d\alpha}{dt} - \frac{1}{4} \alpha^2.$$

The invariant may be obtained following the orthogonal functions procedure [31], whereby two linearly independent solutions of (10) are considered, say $q_1, q_2$. The above equation is evaluated for the solution $q_2$ multiplied throughout by $q_1$. The equation is also evaluated for $q_1$ times $q_2$ and the difference between the two expressions is assessed

$$q_1 \frac{d^2 q_2}{dt^2} - q_2 \frac{d^2 q_1}{dt^2} + \alpha \left( q_1 \frac{dq_2}{dt} - q_2 \frac{dq_1}{dt} \right) = 0.$$

The difference involving the second derivatives may be written as

$$\frac{d}{dt} \left( q_1 \frac{dq_2}{dt} + q_2 \frac{dq_1}{dt} \right) = \frac{dQ_{12}}{dt};$$

If the two terms are written as the derivative of a product, a first integral may be evaluated.

To this end, cast $\alpha$ in terms of $\mu$ to obtain $\mu \frac{dQ_{12}}{dt} + Q_{12} \frac{d\mu}{dt} = 0$, this expression leads to the invariant

$$W_a = \mu_{init} \exp\left(\int \alpha dt \right) \left( q_1 \frac{dq_2}{dt} + q_2 \frac{dq_1}{dt} \right) = \rho^2 \gamma \mu_{init} \exp\left(\int \alpha dt \right);$$

(11)

Consider as an example the constant absorption case $\alpha = \text{const}$. The solution is then $\kappa = A_0 \exp\left( \pm i \Omega \sqrt{\mu} t \right) = A_0 \exp\left( \pm i \sqrt{\Omega^2 - \frac{1}{4} \alpha^2} t \right)$ and therefore $q = \kappa \mu^{-\frac{1}{2}}$. The solution
to the harmonic oscillator equation with absorption (10) with constant coefficients is given by

\[ q = \rho \exp (i \gamma) = A_0 \mu \exp \left( -\frac{\alpha^2}{2} t \right) \exp \left( \pm i \sqrt{\Omega^2 - \frac{1}{4} \alpha^2} t \right). \]  

(12)

The invariant (11) for constant frequency and absorption is then

\[ W_a = \mu Q_{12} = A_0^2 \sqrt{\Omega^2 - \frac{1}{4} \alpha^2}, \Omega, \alpha = \text{constant}. \]  

(13)

This result is valid provided that \( 4\Omega^2 \geq \alpha^2 \). If the opposite condition is fulfilled i.e. \( 4\Omega^2 < \alpha^2 \), the exponentials in the solution for \( q \) are real and the phase \( \gamma \) (as well as its derivative) is then zero. The invariant from (11) is therefore zero for \( 4\Omega^2 < \alpha^2 \).

### 2.2.3 Time dependent mass

The absorption problem is formally equivalent to the harmonic oscillator equation with variable mass since

\[ \frac{dp}{dt} + k^2 q = 0 \Rightarrow \frac{d^2q}{dt^2} + \frac{k}{M} \frac{dM}{dt} \frac{dq}{dt} + \Omega^2 q = 0, \]  

where \( \Omega = \sqrt{k/M} \).

However, for certain functions such as a step mass dependence, the derivatives involved in obtaining \( \alpha \) and \( \frac{d\alpha}{dt} \) produce divergences that complicate the ODE solutions.

Alternatively, a Darboux type transformation may be invoked to avoid this problem [32]. The TDHO equation \( \frac{d^2\psi}{dt^2} + \Omega^2 \psi = 0 \) is solved for a parameter \( \Omega = \sqrt{k/M} \). A step dependence in the mass \( M(t) \) again produces a step function in \( \Omega \) but no nasty divergences. The derivative of the TDHO equation is \( \frac{d^3\psi}{dt^3} + 2\Omega \frac{d\Omega}{dt} \psi + \Omega^2 \frac{d\psi}{dt} = 0 \). Substitution with \( q = \frac{d\psi}{dt} \) and \( \frac{d\Omega}{dt} = \frac{1}{2} \sqrt{\frac{M}{k} \frac{1}{M^2}} \left( Mk - kM^2 \right) \) recreates the time dependent mass ODE \( \frac{d^2q}{dt^2} + \left( \frac{M}{M} - \frac{k}{k} \right) \frac{dq}{dt} + \Omega^2 q = 0 \). Therefore, the solution \( q \) of the time dependent mass problem can be obtained from the derivative of a function \( \psi \) that is solution of a frequency dependent harmonic oscillator. However, the time dependent function \( \Omega \) is not rescaled as in the previous variable transformation case where \( \Omega \to \Omega \kappa \).

### 2.3 Nonlinear superposition principle

The nonlinear amplitude superposition principle arises from the translation of the linear TDHO superposition principle to the superposition of solutions satisfying the nonlinear amplitude equation.

Given particular amplitude and phase solutions \( A \) and \( s \), the general solution may be constructed from the nonlinear superposition principle [14]. Consider a general complex valued solution of the form

\[ \tilde{q}_g = \rho e^{i \gamma} = ae^{is} + be^{-is}. \]  

(14)

where \( a \) and \( b \) are solutions to the nonlinear amplitude equation (6) and the derivative of \( s \) is a solution to the nonlinear frequency equation (8). The two addends correspond to opposite
phase solutions $\pm s$. The general solution in terms of the particular opposite phase solutions is a statement of the nonlinear superposition principle.

$$\rho = \sqrt{a^2 + b^2 + 2ab \cos(2s)}$$  \hspace{1em} (15)

and phase functions

$$\gamma = \arctan \left[ \frac{a - b}{a + b} \tan(s) \right].$$  \hspace{1em} (16)

The Wronskian or orthogonal functions invariant (7) is then

$$W = (a^2 - b^2) \dot{s} + (b\dot{a} - a\dot{b}) \sin(2s).$$  \hspace{1em} (17)

**Nonlinear amplitude superposition principle.** Given two particular solutions $a, b$ to the nonlinear amplitude equation $W^2 - \rho^3 (\ddot{\rho} + \rho \Omega^2) = 0$, the general solution is given by $\rho = \sqrt{a^2 + b^2 + 2ab \cos(2s)}$, where $s$ is given by $W = (a^2 - b^2) \dot{s} + (b\dot{a} - a\dot{b}) \sin(2s)$.

**Nonlinear phase superposition principle.** Given two opposite phase solutions with frequencies $+\dot{s} = \omega, -\dot{s} = -\omega$ to the nonlinear phase equation $\gamma \frac{d^3 \gamma}{dt^3} - \frac{3}{2} \gamma^2 + 2 [\dot{\gamma}^2 - \Omega^2] \dot{\gamma} = 0$, the general solution is given by $\gamma = \arctan \left[ \frac{a - b}{a + b} \tan(s) \right]$. In frequency terms, the general solution to the differential equation $\omega \ddot{\omega} - \frac{3}{2} \omega^2 + 2 [\omega^2 - \Omega^2] \omega^2 = 0$ is given by $\omega_y = \frac{(a^2 - b^2) \omega}{a^2 + b^2 + 2ab \cos(2\int \omega dt)}$.

### 2.4 Novel solutions

In the slowly varying time parameter regime, solutions fulfilling adiabatic invariance have been successfully used since the development of statistical mechanics [33, p. 157]. In contrast, problems involving time dependent parameters of the order of the fundamental oscillation of the system or faster need to be tackled using different techniques such as exact invariants. The sub period regime is defined for a the time dependent parameter that varies appreciably in an interval much shorter than the fundamental period of the system although not necessarily infinitesimal. A step function that varies abruptly in time is the fast limit of the sub-period regime. The relevance of this problem in many areas of physics has been pointed out before [34]. The step function limit is usually tackled through piecewise integration of the appropriate equations in different intervals. The solutions at each interval are then brought together via continuity conditions at the interfaces. For example, a linear sweep time dependence has been solved using this procedure [35]. However, the derivative of the time dependent parameter in this approach is not continuous at the interfaces.

In the present approach, approximate analytical solutions are proposed for monotonic functions whose variation takes place in the sub-period regime. To this end, consider the form given by the general solution (14) but propose a solution where the time dependent functions $a, b$ on their own, are no longer solutions to the nonlinear amplitude differential
Consider a time dependent parameter that is constant for times \( t_1 \ll t_s \) or \( t_2 \gg t_s \) and a monotonic increasing (or decreasing) function in the vicinity of \( t_s \). Allow for an initial condition at time \( t_1 \) where the amplitude \( \rho = a_1 \) \((a = a_1, b = 0)\) and a final condition where \( a = a_2, b = b_2 \). Fix the phase equal to zero at \( t_s \); Furthermore, let the phase be given by
\[
s = \int_{t_s}^{t} \Omega \left( t' \right) dt',
\] (18)
where the lower limit ensures that the phase is zero at \( t_s \). The invariant relationship (17) then yields
\[
W = (a^2 - b^2) \Omega.
\] (19)
The sub-period condition restricts the parameter to vary rapidly in a time much shorter than the characteristic period of the system but is otherwise arbitrary. The invariant (19) holds for the entire time domain since the second term in (17) is negligible far from \( t_s \) because the amplitudes \( a, b \) are then constant and in the vicinity of \( t_s \) because the phase is insignificant.

The continuity of \( q \) at \( t_s \) and \( t_s - \Delta \) together with the sub-period approximation \((s \text{ constant in this interval})\) imposes the condition \( a_1 = a + b \). These two conditions, the invariant with sub-period restriction and continuity of \( q \), yield amplitudes of the form
\[
a = \frac{1}{2} a_1 \left( 1 + \frac{W}{a_1^2 \Omega} \right), \quad b = \frac{1}{2} a_1 \left( 1 - \frac{W}{a_1^2 \Omega} \right),
\] (20)
where it should be recalled that neither of them are solutions to the nonlinear amplitude equation. However, their nonlinear superposition according to (15)
\[
\rho = \frac{a_1}{\sqrt{2}} \sqrt{1 + \frac{\Omega^2}{\Omega^2} + \left( 1 - \frac{\Omega^2}{\Omega^2} \right) \cos \left( 2 \int_{t_s}^{t} \Omega \left( t' \right) dt' \right)},
\] (21)
is an approximate solution to the nonlinear amplitude equation. The initial condition \( W = a_1^2 \Omega_1 \) has been invoked. The explicit time dependence of the parameter may be chosen from a variety of functions provided that it is monotonic and fulfills the sub-period criterion. A hyperbolic tangent dependence has been used in computer evaluations
\[
\Omega = \omega_0 \left[ 1 + \frac{\Delta}{2} \left( 1 + \tanh \left[ \epsilon \left( t - t_s \right) \right] \right) \right],
\] (22)
where \( \Delta \) is the height of the step and \( \epsilon \) is a measure of the slope at \( t_s \). The comparison of numerical solutions to the nonlinear amplitude equation and the analytical solution show very good agreement [36] as may be seen from figure 1.

In a similar fashion the derivative of the phase \( s \) given by (18) is not a solution to the frequency equation (8). Nonetheless, the nonlinear superposition (16) of opposite phases
\[
\gamma = \arctan \left[ \frac{a - b}{a + b} \tan s \left( t \right) \right]
\] is an approximate solution to the nonlinear phase equation. The solution for the phase is then
\[
\gamma = \arctan \left[ \frac{\Omega_1}{\Omega} \tan \int_{t_s}^{t} \Omega \left( t' \right) dt' \right] + (t_s - t_0) \Omega_1
\] (23)
Figure 1: $\rho$ vs. $t$ [upper left inset] - Nonlinear amplitude differential equation numerical solution (6) in solid line and analytic approximation (21) in dashed line; Time dependent parameter $\Omega$ vs. $t$ [lower left inset] from eq. (22) with $\Delta = 1$, $\epsilon = 20$ and $t_s = 2$. The initial conditions are $\rho(t = -\infty) = 1$, $\dot{\rho}(t = -\infty) = 0$. Detailed portion [upper right inset] of $\rho$ vs. $t$ numerical and analytic solutions in the region where the parameter varies rapidly.

and for the frequency $\omega = \dot{\gamma}$

$$\omega = \frac{\Omega_1 \left[1 - \frac{1}{2t_s^2} \frac{d\Omega}{dt} \sin \left(2 \int_{t_s}^t \Omega(t') \, dt'\right)\right]}{\left(\frac{t_s}{2t} \right)^2 \sin^2 \left(\int_{t_s}^t \Omega(t') \, dt'\right) + \cos^2 \left(\int_{t_s}^t \Omega(t') \, dt'\right)}.$$  

(24)

The approximate results for a sub-period parameter are exact for the step function when $\epsilon \to \infty$. There is an errata in ref. [36, p.193, eq. (24)], where the second term in the numerator of (24) is missing; Due to the small phase condition in the vicinity where $\Omega$ varies appreciable, this term does not affect the results presented therein. These solutions, besides a monotonic sub-period parameter, require an initial condition where the amplitude is constant. Whether this condition can be lifted is a matter that is being evaluated.

These solutions have been recently used in molecular biochemistry to display the conformational transition from the right-handed B-DNA helix to the left-handed Z-DNA helix. The model proposed by Lim [24] involves a TDHO equation with a hyperbolic secant function parameter. In this case, the solutions are adequate although the monotonic requirement is not fulfilled. It is likely that the monotonic restriction can be lifted provided that the less restrictive sub-period condition is enforced. These solutions have also been employed to
describe the evolution of minimum uncertainty states in quantum mechanics as we shall see in section 3.3.1.

The TDHO real solution \( q = \rho \cos \gamma \) has been expressed in terms of the amplitude \( \rho \) that is dependent on the phase \( s \) together with the argument of the cosine function that involves a nonlinear phase \( \gamma \). However, it may also be written in terms of an amplitude \( G \) independent of the phase \( s \) and a trigonometric function with a linear phase \( \Gamma \). To this end, a constant phase \( \theta \) is introduced in both terms of the general solution (14). The variables \( s \) and \( \theta \) can then be interchanged without altering the real part of the solution \( q = G \cos \Gamma \). This fact can be used to obtain the desired solutions. The phase dependent amplitude \( \rho \) is then mapped into an \( s \) independent amplitude \( \rho \rightarrow G = \sqrt{a^2 + b^2 + 2ab \cos (2\theta)} \). For a sub period function with opposite phase amplitudes given by (20), the amplitude is

\[
G = \frac{a_1}{\sqrt{2}} \sqrt{1 + \left( \frac{\Omega^2_1}{\Omega^2_2} \right) + \left( 1 - \frac{\Omega^2_1}{\Omega^2_2} \right) \cos (2(t_s - t_0) \Omega_1),}
\]

where the difference \( t_s - t_0 \) establishes the phase when the parameter variation takes place. The amplitude \( G \) is then time independent for constant \( \Omega \neq \Omega_1 \) even if this quantity differs from \( \Omega_1 \). It is worthwhile noticing that although we requested the phase dependent amplitude \( \rho \) to be continuous, the amplitude \( G \) is not continuous in the limit of an abrupt step.

The nonlinear phase \( \gamma \) is correspondingly mapped into a function linearly dependent on \( s \), namely, \( \gamma \rightarrow \Gamma = s + \arctan \left[ \frac{a-b}{a+b} \tan \theta \right] \). In a steep function case

\[
\Gamma (t) = \int_{t_s}^{t} \Omega (t) \, dt + \arctan \left\{ \frac{\Omega_1}{\Omega (t)} \tan \left[ (t_s - t_0) \Omega_1 \right] \right\}.
\]

The result is to be expected because the solution of the TDHO equation (1) in a constant parameter region is sinusoidal. In a constant parameter region, the amplitude \( G \) is time independent and the phase \( \Gamma \) has a linear dependence on time. However, the quantity \( G^2 \Gamma \) is not constant in contrast with the invariant \( \rho^2 \gamma \). The ratio of the amplitudes long before and after the parameter variation lies between the limits

\[
1 \leq \frac{G_2}{G_1} \leq \frac{\Omega_1}{\Omega_2},
\]

depending on whether the abrupt variation takes place at maximum or zero displacement. The condition \( \Omega_2 > \Omega_1 \), reflects an increasing oscillator stiffness. If \( \Omega_2 < \Omega_1 \), the stiffness decreases; It should be clear that if \( \Omega_2 \) decreases to zero the amplitude diverges since the motion is no longer bounded.

The solutions obtained by the traditional piecewise integration method in the abrupt limit are in fact of the form \( q = G \cos \Gamma \) and thus do not preserve the invariant relationship. The reason being that two amplitude equations (6) are being solved with different constants \( W \) for each time span. In contrast, the analytic procedure procedure presented here solves approximately the non-autonomous nonlinear differential equation and preserves the invariant. The analytic solution becomes exact in the limit of an abrupt change in the time dependent parameter.
2.5 Indeterminacy

The amplitude and phase functions in the preceding section may be given by the pair (25, 26) or the pair (21, 23) since \( \Re \{q\} = \rho \cos \gamma = G \cos \Gamma \). The indeterminacy in these variables exhibited by these two results is in fact a particular example of a more general result. The trigonometric identity \( \cos [\arctan (\alpha/\beta)] = \beta/\sqrt{\alpha^2 + \beta^2} \), may be used to obtain[37]

\[
\begin{align*}
\text{amplitude} & \quad \text{cos} \phi = \sqrt{A^2 \cos^2 \phi + B^2 \sin^2 \phi} \cos \left[ \arctan \left( \frac{B}{A} \tan \phi \right) \right] \\
\text{phase} & \quad \text{amplitude} \\
\end{align*}
\]

(27)

for any real function \( B \). The amplitude function is then

\[
\rho = \sqrt{A^2 \cos^2 \phi + B^2 \sin^2 \phi},
\]

(28)

whereas the phase is

\[
\gamma = \arctan \left( \frac{B}{A} \tan \phi \right).
\]

(29)

If \( A \) satisfies the amplitude differential equation, the amplitude \( \rho \) is also a solution if \( B \) satisfies the amplitude equation (6) consistent with the nonlinear superposition principle described in section 2.3. The amplitude coefficients for the real and imaginary solutions \( A, B \) are related to the opposite phase amplitudes \( a, b \) by \( A = a + b, B = a - b \). Therefore, if we cast the coordinate solution in terms of amplitude and phase variables, these variables are indeterminate up to a particular nonlinear amplitude solution \( B \) according to equations (28) and (29).

Let us address the consequences of this indeterminacy concerning two issues, conserved quantities and measurement theory.

2.5.1 Energy

The orthogonal functions invariant (17) in terms of the variables \( A, B \) is

\[
W = \rho^2 \dot{\gamma} = AB \dot{\phi} + \frac{1}{2} \left( A \ddot{B} - B \ddot{A} \right) \sin (2\phi).
\]

(30)

The value of the invariant depends on the choice of \( B \) and clearly vanishes if \( B \) is set to zero. If the invariant is fixed to a specific value, the indeterminacy is no longer present since the function \( B \) may be obtained from (30). It is worth mentioning that equation (17) is nonlinear in \( a, b \) and \( b \) cannot be readily integrated. However, (30) is linear in \( A, B \) so that it may be integrated to find \( B \) in terms of the particular amplitude and phase variables

\[
B = A \cot \phi \left( c_1 + W \int_1^t \frac{\sec^2 \phi (t_i)}{A(t_i)^2} dt_i \right).
\]

(31)
If the quantities $A, B$ are time independent, the invariant (30) simplifies to $W = A_0 B_0 \omega_0$. The eigenvalue of the invariant clearly differs for different values of $B_0$.

In contrast, if there is a time varying parameter, the Hamiltonian is then time dependent and it no longer represents the energy of the system. In the particular case of constant parameter $k = \Omega^2 m$, the energy of an oscillator is given by the sum of the kinetic and potential energies $E = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2$. In terms of amplitude and phase variables

\[ E = \frac{1}{2} m (\dot{\rho} \cos \gamma - \rho \dot{\gamma} \sin \gamma)^2 + \frac{1}{2} k \rho^2 \cos^2 \gamma. \]  

(32)

A constant parameter $\Omega = \Omega_0$ requires that $A = A_0, B = B_0$. Substitution of $\rho$ and $\gamma$ from (28) and (29) gives $E = \frac{1}{2} m A_0^2 (\dot{\phi} \sin \phi)^2 + \frac{1}{2} k \rho^2 \phi$. Notice that although (32) involves $B_0$ through the amplitude $\rho$ and phase $\gamma$ functions, this last expression is no longer dependent on the constant $B_0$. Substitution of $\sqrt{k/m} = \dot{\phi} = \Omega_0$, yields the well known result $E = \frac{1}{2} m A_0^2 \Omega_0^2$. Therefore, the energy function is identical for either $A_0 \cos (\Omega_0 t + \theta)$ or $\rho(t) \cos \gamma(t)$ with arbitrary $B_0$ provided that we calculate it from the sum of kinetic and potential energies.

2.5.2 Measurement

The amplitude and phase of harmonic oscillatory motion are measured with the following underlying assumptions:

- The amplitude $A_0$ is set equal to the maximum displacement from equilibrium.
- The amplitude $A_0$ is considered to be constant if the system attains the same maximum displacement after each cycle.
- In order to evaluate the frequency, the motion is tagged at a given position and velocity. One oscillation is completed when the body or disturbance returns to the same position and velocity coordinates.
- The frequency is considered constant if the period of the oscillations is time independent.
- The phase, linear in time, is then considered to be equal to the frequency multiplied by time with an arbitrary initial phase.

However, the present results show that if the maximum displacement is constant it does not necessarily follows that the amplitude is time independent. It is equally valid to consider that the amplitude is time dependent according to (28) with $A, B$ constants. On the other hand, even if the period of oscillation is constant, the phase may be non-linear and the frequency time dependent according to (29) with $\phi = \omega t$.

It may be argued, with good reasons, that since the constant amplitude and linear phase are equally valid they should be preferred since they are a simpler choice. Nonetheless, it
is not always possible to make such a choice without incurring into inconsistencies [37].

Such is the case of a system initially with constant amplitude and frequency governed by
the oscillator equation with constant parameter $\Omega_1^2 (t \ll t_s) = \Omega_1^2$, which, subsequently
undergoes a non adiabatic parameter change. The final state necessarily involves a time
dependent amplitude and non-linear phase if the invariant condition is fulfilled. This result
may be seen from the approximate analytical solution described in section 2.4. A system
with initial parameter $\Omega_1$, from the solution (21) has a constant amplitude $\rho_1 = a_1$. In the
final state with constant parameter $\Omega_2$, the amplitude $\rho (t)$ is inevitably time dependent if
$\Omega_2 \neq \Omega_1$.

The trajectory is identical either for the constant amplitude - linear phase or the nonlin-
ear amplitude and phase solutions. Therefore they are indistinguishable. The same is true
for the energy. However, the orthogonal functions invariant is different and the question
is whether it is measurable. In the present non propagating problem the invariant may be
arbitrarily set at a given initial condition. Nonetheless, as we shall see in subsection 4.1, in
propagating phenomena the two situations may be perceived as distinct.

### 2.6 Adiabatic and exact invariants

Adiabatic invariants require that the time evolution of the parameters should be slow com-
pared with the characteristic period of the time independent parameters [33]. The adiabatic
invariant for a system executing periodic motion with slowly varying parameters is given by
the area integral in phase space $I_{\text{adiab}} = (1/2\pi) \int \int dpdq$, where $q$ is the canonical position
coordinate and $p$ its corresponding momentum. In particular, the adiabatic invariant for a
one dimensional time dependent oscillator is equal to the ratio of its energy over frequency

$$ I_{\text{adiab}} = \frac{\mathcal{E}}{\omega}. \tag{33} $$

This result was first derived by Ehrenfest [38] in relation with Planck’s oscillators. Einstein
and Bohr discussed this constant ratio for a pendulum with slowly varying length at Solvay
[39]. The adiabatic approach has been used extensively in the averaging of Hamiltonian
systems in statistical mechanics.

The energy of an oscillator in the adiabatic condition is generalized from the time inde-
pendent case to a time dependent amplitude and frequency

$$ \mathcal{E} = \frac{1}{2} m A^2 \Omega_0^2 \rightarrow \mathcal{E}_{\text{adiab}} \approx \frac{1}{2} m A^2 (t) \Omega^2 (t), \tag{34} $$

where $\hat{\Omega} \ll \Omega$. From the energy expression in amplitude and phase variables (32) the
approximations i) $\frac{\dot{\rho}}{\rho} \ll \dot{\gamma}$ and ii) $\dot{\gamma}^2 = \frac{k}{m}$ reduce the energy to $\frac{1}{2} m A^2 \Omega^2 = \frac{1}{2} m \rho^2 \omega^2$. On the other hand, the orthogonal functions invariant in amplitude and phase variables (7) may
be rewritten as

$$ \frac{1}{2} m W = \frac{1}{2} m \rho^2 \dot{\gamma}^2 \dot{\gamma} = \frac{1}{2} m \rho^2 \omega^2 \tag{35}. $$
The exact invariant may thus be approximately written as the ratio of energy over frequency. The first estimate i) is a slowly varying envelope type approximation. The second condition ii) implies that first and second order derivatives in the nonlinear frequency equation (8) are ignored so that the oscillator frequency \( \omega \) is equal to the time varying parameter \( \Omega \). A nice feature of the orthogonal functions invariant, in contrast with the Ermakov-Lewis invariant, is that it coincides with the adiabatic invariant under the above mentioned approximations.

Adiabatic processes require that the time dependent parameter \( \Omega \) varies slowly with respect to time. This condition in turn, implies that the amplitude and frequency must be quasi-static functions. Therefore, if we invert the reasoning, time dependent amplitude and frequency functions should be associated with the non-adiabatic regime if the initial amplitude is time independent. Thus, harmonic oscillator systems with fast time dependent parameters with respect to the period of the system, i.e. non-adiabatic, must exhibit rapid variations in the amplitude and frequency variables. Whether this assertion may be extended to other dynamical systems is still an open issue.

3 Quantum domain

The extension of the theory of invariants to the quantum realm has evolved in, at least, two directions. On the one hand, the one dimensional time independent Schrödinger equation is formally equivalent to the TDHO equation. The equivalence requires i) the substitution of temporal variable by the spatial coordinate; ii) A constant shift of the potential \( V(x) \) by the energy eigenfunction with the appropriate scaling that represents the spatially (formerly time) dependent parameter \( \Omega^2(x) \rightarrow \frac{2m}{\hbar^2} (E - V(x)) \). The results obtained in classical invariant theory are thus applicable for spatially arbitrary time independent potentials in stationary one dimensional quantum theory.

On the other hand, quantum mechanical expressions of the classical invariant operators have been used in order to obtain exact solutions to the time dependent Schrödinger equation. To this end, the classical Hamiltonian is translated into a quantum Hamiltonian by considering the canonical coordinate and momentum as time independent operators obeying the commutation relationship \( [\hat{q}, \hat{p}] \equiv \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar \). The quantum treatment then becomes a 1+1 dimensional problem where the wave function depends on a spatial as well as the temporal variable. A potential with an arbitrary time dependence is identified with the coordinate operator of the Hamiltonian. Exact invariants have been derived to tackle a limited class of admissible potentials [39]. The most relevant cases are the quadratic spatial dependence which leads to the quantum mechanical time dependent harmonic oscillator (QM-TDHO) and the linear potential [40].

3.1 Solution to the Schrödinger equation

The QM-TDHO has been solved under various circumstances such as damping and a time dependent mass. This problem was originally worked out in terms of time-dependent Green functions using a path integral method. Other techniques have been used in order to solve
the corresponding time dependent Schrödinger equation such as the time-space re-scaling or transformation method and the time dependent Ermakov-Lewis invariant method [25]. The solutions thus obtained are functions of a $c$-number quantity whose differential equation needs to be solved.

The derivation reproduced here, in contrast with previous accounts that invoke the Lewis invariant, considers the orthogonal functions quantum invariant as the starting point. This invariant is a quantum version of the classical orthogonal functions invariant also referred in the literature as a linear integral of motion operator [41]. Unitary transformations are then applied to the invariant in order to obtain an explicitly time independent quantity. The transformations map the orthogonal functions quantum invariant directly onto the momentum operator. Once these transformations are established, the wave function is transformed in order to obtain a simplified Schrödinger equation which is readily integrable.

3.1.1 Quantum invariant

The time dependent Hamiltonian of the QM-TDHO is

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \Omega^2 \hat{q}^2).$$

(36)

In order to translate the classical invariant (2) to the quantum domain, let the function $q_2$ and its time derivative become the quantum coordinate and momentum operators. The linearly independent solution $q_1$ is then treated as a real $c$-number, which obeys the classical TDHO equation. The orthogonal functions quantum invariant operator is then

$$\hat{G}_1 = q_1 \hat{p} - \dot{q}_1 \hat{q},$$

(37)

where the dot denotes differentiation with respect to time. The invariance of this linear integral of motion may be corroborated by direct evaluation of its time derivative. The partial time derivative reads $\frac{\partial \hat{G}_1}{\partial t} = \dot{q}_1 \hat{p} - \ddot{q}_1 \hat{q},$ this quantity does not vanish since the invariant is explicitly time dependent. However, its total time derivative is indeed zero

$$\frac{d\hat{G}_1}{dt} = \frac{\partial \hat{G}_1}{\partial t} - \frac{i}{\hbar} [\hat{G}_1, \hat{H}] = - (\ddot{q}_1 + \Omega^2 q_1) \hat{q} = 0,$$

since $[\hat{G}_1, \hat{H}] = \frac{1}{2} \dot{q}_1 (\hat{p}^2 \hat{q} - \hat{q} \hat{p}^2) - \frac{1}{2} \Omega^2 q_1 (\hat{p} \hat{q}^2 - \hat{q}^2 \hat{p}) = -i\hbar (\ddot{q}_1 \hat{p} + \Omega^2 q_1 \hat{q}).$ The obtention of a second invariant given a first invariant has been subject of several communications [42], [43]. It is worth remarking that the existence of a second invariant warrants complete integrability for a Hamiltonian Ermakov system [44]. Within the present formalism, it is straightforward to introduce a second invariant stemming from the mapping of $q_1$ and $\dot{q}_1$ into the coordinate and momentum operators

$$\hat{G}_2 = -q_2 \hat{p} + \hat{q}_2 \hat{q}.$$  

(38)

These two invariants obey the commutation relation

$$[\hat{G}_1, \hat{G}_2] = (-q_1 q_2 + q_2 \dot{q}_1) (\hat{p} \hat{q} - \hat{q} \hat{p}) = -i\hbar W$$

(39)
where \( W \) is the constant Wronskian defined by Eq. (2). The quantum invariants (37) and (38) are linear in the coordinate variables whereas the classical invariant (2) involves a quadratic form. The reason being that, in the quantum version, one of the functions is being mapped onto operators. The classical quadratic form is directly related to the Hamiltonian of the system. However, in the quantum case, it is necessary to take the product of two linear invariants in order to produce a quantum quadratic form.

### 3.1.2 Unitary transformations

The goal of the transformations in this context is to map the invariant into an explicitly time independent quantity. The transformed “invariant” is then no longer time independent although its partial time derivative does vanish. Such a transformation leads to the solution of the Schrödinger equation as we shall demonstrate in the following sections.

Unitary transformations of the form

\[
\hat{T}_u = \exp \left( i \frac{\ln \alpha}{2 \hbar} \left( \hat{q} \hat{p} + \hat{p} \hat{q} \right) \right) \exp \left( -i \frac{\dot{\alpha}}{2 \hbar} \hat{q}^2 \right),
\]

(40)

that may be written in the nicely symmetrical form

\[
\hat{T}_u = \exp \left( i \frac{\ln \alpha}{2 \hbar} \frac{d\hat{q}^2}{dt} \right) \exp \left( -i \frac{\dot{\alpha}}{2 \hbar} \hat{q}^2 \right).
\]

The time independent operator in the propagator (43) is, as we shall presently show, an invariant that is not unique [46]. On the one hand, it may be proportional to the square of the orthogonal functions invariant operator (with \( q_1 = u \) for subindices economy)

\[
\hat{I}_u = \frac{1}{2} \hat{G}_1^2 = \frac{1}{2} \left( u\hat{p} - \dot{u} \hat{q} \right)^2,
\]

(41)

where the function \( \alpha \to u \in \mathbb{R} \), replaced in the invariant as well as in the transformation expressions, obeys the TDHO equation (1). On the other hand, the propagator may be written using the Ermakov Lewis invariant with \( \alpha \to \rho \), where \( \rho \) obeys the nonlinear amplitude equation (6). In either case, it is seen that the invariant in the time dependent case enters the propagator expression in an analogous fashion as the Hamiltonian does in the time independent case.

The orthogonal functions quantum invariant (37) is then transformed onto the momentum operator \( \hat{G}_1' = \hat{T}_u \hat{G}_1 \hat{T}_u^\dagger = \hat{p} \), since the displacement operator maps \( \hat{G}_1 = u\hat{p} - \dot{u} \hat{q} \to u\hat{p} \) and the scaling operator maps \( u\hat{p} \to \hat{p} \). Unitary transformations in quantum mechanics
correspond to canonical transformations in classical mechanics [47]. In this case, the transformed invariant becomes the new momentum just as the canonical transformation of the adiabatic invariant turns it into the action variable [33]. However, in the present case the invariant is exact since it has been established without invoking the adiabatic approximation.

### 3.1.3 Solution in terms of the invariant

The solution to the Schrödinger equation

\[ i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t) |\psi(t)\rangle \]  

(42)

for the QM-TDHO Hamiltonian given in Eq. (36) may then be written in terms of a propagator that involves a time independent operator together with an appropriate transformation of the wave function \( |\psi(t)\rangle = \hat{U}_\alpha \hat{T}_\alpha(0) |\psi(0)\rangle \). The propagator is given by

\[
\hat{U}_\alpha = \exp \left( -is_\alpha \hat{I}_\alpha \right); \quad s_\alpha \equiv \int_0^t \frac{dt'}{\alpha^2},
\]

(43)

where the function \( \alpha \) is a time dependent \( c \)-number that satisfies the differential equation appropriate to the invariant being used. The quadratic invariant (41) and the Hamiltonian are related by

\[
\hat{I}_\alpha = \alpha^2 \left( \hat{H} - i\hbar \frac{\partial \hat{T}_\alpha^\dagger}{\partial t} \hat{T}_\alpha \right),
\]

(44)

whereas the transformed invariant in terms of the Hamiltonian is

\[
\hat{I}'_\alpha = \hat{T}_\alpha \hat{I}_\alpha \hat{T}_\alpha^\dagger = \alpha^2 \left( \hat{T}_\alpha \hat{H} \hat{T}_\alpha^\dagger - i\hbar \hat{T}_\alpha \frac{\partial \hat{T}_\alpha^\dagger}{\partial t} \right).
\]

(45)

To resume, the invariant operators \( \hat{G}_{1,2}, \hat{I}_\alpha \) are time independent although their partial time derivatives are different from zero. On the other hand, the transformed operators \( \hat{G}'_{1,2}, \hat{I}'_\alpha \) are implicitly time dependent but their partial time derivative is zero.

### Example

In terms of the transformed wave function \( |\phi_u(t)\rangle = \hat{T}_u |\psi(t)\rangle \), the Schrödinger equation (42) is

\[
i\hbar \left( \hat{T}_u^\dagger \frac{\partial |\phi_u(t)\rangle}{\partial t} + \frac{\partial \hat{T}_u^\dagger}{\partial t} |\phi_u(t)\rangle \right) = \hat{H}(t) \hat{T}_u^\dagger |\phi_u(t)\rangle.
\]

Applying the operator \( \hat{T}_u \) from the left leads to an integrable equation of the form

\[
i\hbar \frac{\partial |\phi_u(t)\rangle}{\partial t} = \frac{1}{2} \frac{p^2}{u^2} |\phi_u(t)\rangle.
\]
The solution to this equation is

\[ |\phi_u(t)\rangle = \exp \left( -\frac{i}{2\hbar} \hat{p}^2 \int_{0}^{t} \frac{dt'}{u^2} \right) |\phi_u(0)\rangle, \]

where \(|\phi_u(0)\rangle\) is the wave function evaluated at the initial time. In terms of the original wave function, the solution to the Schrödinger equation (42) is

\[ |\psi(t)\rangle = \hat{T}_u \exp \left( -\frac{i}{2\hbar} \hat{p}^2 \int_{0}^{t} \frac{dt'}{u^2} \right) \hat{T}_u |\psi(0)\rangle. \tag{46} \]

where \(\hat{T}_u\) is the transformation at time \(t = 0\) and \(u\) fulfills the TDHO equation (1). This solution may be applied to any initial condition \(|\psi(0)\rangle\). It may be easily checked that the above solution is indeed correct by evaluating its partial time derivative to recover the original Schrödinger equation (42). The above example does not necessarily involve the use of the invariant. Nonetheless, in order to choose the appropriate transformation, the guiding principle was to transform the invariant into an explicitly time independent quantity. The solution in terms of the quadratic invariant \(\hat{I}_u\) is

\[ |\psi(t)\rangle = \exp \left( -\frac{i}{\hbar} \hat{I}_u \int_{0}^{t} \frac{dt'}{u^2} \right) \hat{T}_u \hat{T}_u |\psi(0)\rangle. \tag{47} \]

In a similar fashion, the the solution in terms of the quantum Ermakov-Lewis invariant (50) is

\[ |\psi(t)\rangle = \exp \left( -\frac{i}{\hbar} \hat{I}_\rho \int_{0}^{t} \frac{dt'}{\rho^2} \right) \hat{T}_\rho \hat{T}_\rho |\psi(0)\rangle. \tag{48} \]

where \(\rho\) satisfies the nonlinear amplitude equation (6).

### 3.2 Amplitude and phase discrete states

The quantum invariant formalism can be mapped from the coordinate and momentum operators into an invariant in terms of amplitude and phase operators. However, these operators as shown below, are not unique. It is of particular interest to establish the coherent states that reduce to the corresponding variables in the classical limit.

From the squared sum of the orthogonal functions quantum invariants (37) and (38), we may construct the quadratic invariant operator

\[ \hat{I}_\rho = \frac{1}{2} \left( \hat{G}_1^2 + \hat{G}_2^2 \right), \tag{49} \]

which, in terms of the position and momentum operators is \(\hat{G}_1^2 + \hat{G}_2^2 = (q_1^2 + q_2^2)p^2 + (\dot{q}_1^2 + \dot{q}_2^2)\dot{q}^2 - (q_1\dot{q}_1 + q_2\dot{q}_2)(\dot{p} + \dot{q})\). This expression may be rewritten as a function of the amplitude function (3) by noticing that \(\dot{\rho} \rho = q_1\dot{q}_1 + q_2\dot{q}_2\) and that the orthogonal functions obey (2), so that

\[ \frac{W^2}{\rho^2} + \rho^2 = \frac{(q_1\dot{q}_2 - q_2\dot{q}_1)^2}{\rho^2} + \frac{(q_1\dot{q}_1 + q_2\dot{q}_2)^2}{\rho^2} = (\dot{q}_1^2 + \dot{q}_2^2). \]
The operator in terms of the amplitude function $\rho$ is then

$$I_\rho = \frac{1}{2} \left[ \left( \frac{W \dot{q}}{\rho} \right)^2 + (\rho \dot{p} - \dot{\rho} \dot{q})^2 \right]. \quad (50)$$

This is the Ermakov Lewis quantum invariant where the real constant $W$ is usually normalized but in general may differ from one [48]. The above procedure is a simple derivation of the quantum Ermakov Lewis invariant, which has otherwise been obtained using more complex mathematical methods [25]. The non Hermitian linear invariant $\hat{I}_c$ introduced by Malkin et al. [49, 50] written in terms of the Hermitian orthogonal functions quantum invariant is

$$\hat{I}_c = \hat{G}_1 - i\hat{G}_2.$$

An operator that can be written as the sum of two squares may be expressed in terms of two adjoint complex quantities. To wit, given an operator $\hat{\beta}$ that can be expressed as

$$\hat{\beta} = \hat{b}_1^2 + \hat{b}_2^2,$$

provided $[\hat{b}_1, \hat{b}_2] = c$, with $c$ a classical-number; There exist annihilation and creation operators $\hat{b} = \hat{b}_1 + i\hat{b}_2$, $\hat{b}^\dagger = \hat{b}_1 - i\hat{b}_2$ so that the operator may be written as $\hat{\beta} = \hat{b}^\dagger \hat{b} - i [\hat{b}_1, \hat{b}_2]$. For instance, annihilation and creation operators for the Hamiltonian (36) may be written as [51]

$$\hat{B} = \frac{1}{\sqrt{2}} \left( \Omega^{1/2} \dot{q} + i\dot{p}/\Omega^{1/2} \right), \quad \hat{B}^\dagger = \frac{1}{\sqrt{2}} \left( \Omega^{1/2} \dot{q} - i\dot{p}/\Omega^{1/2} \right).$$

However, the way in which the $\hat{\beta}$ operator is written as the sum of two squares need not be unique. In fact, for the invariant operator $\hat{I}_\rho$, expressions (49) and (50) are two such possibilities. The former leads to annihilation and creation operators of the form

$$\hat{A} = \frac{1}{\sqrt{2}} \left( \hat{G}_1 - i\hat{G}_2 \right), \quad \hat{A}^\dagger = \frac{1}{\sqrt{2}} \left( \hat{G}_1 + i\hat{G}_2 \right), \quad (51)$$

where the identification $\hat{b}_1 \rightarrow \hat{G}_1$ and $\hat{b}_2 \rightarrow -\hat{G}_2$ has been made. These operators have also been obtained from the non Hermitian linear invariants that originate from the complex solution of the TDHO equation [41], [52]. The annihilation and creation operators in (51) are also invariant since they are composed by invariant operators. On the other hand, the operators arising from (50) yield

$$\hat{a} (t) = \frac{1}{\sqrt{2}} \left( \frac{\dot{q}}{\rho} + i(\rho \dot{p} - \dot{\rho} \dot{q}) \right), \quad \hat{a}^\dagger (t) = \frac{1}{\sqrt{2}} \left( \frac{\dot{q}}{\rho} - i(\rho \dot{p} - \dot{\rho} \dot{q}) \right). \quad (52)$$

These time dependent annihilation and creation operators were originally introduced by Lewis [12]. The quantum Ermakov invariant in terms of these operators is

$$\hat{I}_\rho = \hat{a}^\dagger (t) \hat{a} (t) + \frac{1}{2} = \hat{A}^\dagger \hat{A} + \frac{1}{2}, \quad (53)$$

where the second equality follows from the definition of this invariant in terms of the orthogonal functions quantum invariants (49). In order to obtain the transformation between the distinct annihilation and creation operators, evaluate

$$\hat{A} e^{-is_\rho} = \frac{1}{\sqrt{2}} \left( \hat{G}_1 - i\hat{G}_2 \right) (q_2 + iq_1).$$
where \( q_1 = -\rho \sin s_\rho, \) \( q_2 = \rho \cos s_\rho. \) The relationship between the orthogonal solutions and their trigonometric representation is not singular. The choice made here represents the function \( q_1 \) leading \( q_2 \) by \( \frac{\pi}{2} \) as \( s_\rho \) increases [53]. Replacing the definitions of the invariants yields
\[
\hat{A} e^{-is_\rho} = \frac{1}{\sqrt{2}} \left( \frac{W \hat{q}}{\rho} + i(\rho \hat{p} - \hat{p} \hat{q}) \right) = \hat{a}.
\]
Therefore the time dependent annihilation (creation) operators may be written as the product of the time independent annihilation (creation) operators times a phase that only involves a \( c \)-number function. This expression may be written as a unitary transformation of a phase shift
\[
\hat{a} = \hat{A} e^{-is_\rho} = \exp \left( is_\rho \hat{I}_\rho \right) \hat{A} \exp \left( -is_\rho \hat{I}_\rho \right). \tag{54}
\]
The equation of motion of this operator is then \( \dot{\hat{a}} = i\omega [\hat{I}_\rho, \hat{a}] \). It is thus seen that the operator \( \omega \hat{I}_\rho \) in the QM-TDHO plays the role that the Hamiltonian does in a time independent harmonic oscillator case [54]. This assertion is consistent with the transformation that relates the invariant and the time dependent Hamiltonian (44) with \( \alpha \rightarrow \rho \) and the frequency \( \omega = 1/\rho^2 \) satisfies the nonlinear frequency equation (8).

### 3.2.1 Turski phase operator

The displacement operator introduced in the previous subsection 3.1.2 can be written in terms of the time dependent annihilation operator (52) as \( \hat{D}(\alpha) = \exp (\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \) where \( \alpha = r \exp(i\theta) \). The vacuum state may then be displaced to obtain a coherent state
\[
|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{55}
\]
The phase operator introduced by Turski [55, 56] is then generalized to the time dependent case
\[
\hat{\Phi} = \int \theta |\alpha\rangle\langle \alpha| d^2\alpha. \tag{56}
\]
This operator obeys the commutation relation \( [\hat{\Phi}, \hat{I}_\rho] = -i \). The time evolution of this operator can be written in terms of the invariant annihilation and creation operators using (54)
\[
\hat{\Phi} = e^{is_\rho \hat{I}_\rho} \left( \int \theta \hat{D}_A(\alpha)|0\rangle\langle 0| \hat{D}_A^\dagger(\alpha) d^2\alpha \right) e^{-is_\rho \hat{I}_\rho}, \tag{57}
\]
where \( \hat{D}_A(\alpha) = \exp (\alpha \hat{A}^\dagger - \alpha^* \hat{A}) \) and the invariant acting over the vacuum state is \( \hat{I}_\rho|0\rangle = \frac{i}{2} |0\rangle \). The time derivative of this expression yields the equation of motion for \( \hat{\Phi} \):
\[
\dot{\hat{\Phi}} = i\omega [\hat{I}_\rho, \hat{\Phi}] = -\omega. \tag{58}
\]
The operator \( \omega \hat{I}_\rho \) once again takes the role of the Hamiltonian since the Turski operator commutes with \( \hat{H} \) in the time independent case. The generalization of the phase to the time
The Nonlinear Amplitude Equation in Harmonic Phenomena

The dependent case is also applicable to other formalisms [57] such as the Susskind-Glogower operators [58]. The coordinate operator from (52) is \( \hat{q} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger) \), and following Dirac [59], the creation and annihilation operators may be written as

\[
\hat{a} = \sqrt{\hat{I}_\rho} e^{-i\Phi}, \quad \hat{a}^\dagger = e^{i\Phi} \sqrt{\hat{I}_\rho}.
\]

(59)

The coordinate operator in amplitude and phase variables is then

\[
\hat{q} = \sqrt{\hat{I}_\rho} e^{-i\Phi} + e^{i\Phi} \sqrt{\hat{I}_\rho} / 2\omega,
\]

(60)

where the amplitude \( \rho \) and phase \( s_\rho \) are identified as \( \rho^2 \rightarrow \hat{I}_\rho/\omega \) and \( s_\rho \rightarrow \hat{\Phi} \). The invariant with the aid of (58) is finally given in amplitude and phase operators as

\[
\hat{I}_\rho = -\frac{\hat{a}^\dagger \hat{a} + 1}{2\omega} \frac{\dot{\hat{\Phi}}}{\hat{\Phi}},
\]

(61)

which has the same structure of the orthogonal functions classical invariant written in amplitude and phase variables (7). The number operator is then identified with \( \hat{n}(t) = \hat{a}^\dagger \hat{a}/\omega \). It should be recalled that this operator is identified with the product of the annihilation and creation pair that arise from the Hamiltonian operator. However, the identification of the number operator in terms of the annihilation - creation pair arising from the invariant is obtained from the correspondence with the orthogonal functions classical invariant. This association may seem dimensionally awkward but it should be remembered that the invariant initial value was normalized to one (50). The explicit introduction of the normalization factor \( \rho_0^2\omega_0 \) makes of course a dimensionless photon number \( \rho^2(t) = \rho_0^2\omega_0/\omega(t) \) for a dimensionless amplitude (usually set to unity \( \rho_0 = 1 \)). The invariant is then

\[
\hat{I}_\rho = -\left( \hat{n}(t) + \frac{1}{2} \frac{\dot{\rho}}{\omega} \right) \hat{\Phi} (t).
\]

(62)

Since \( \hat{a}^\dagger \hat{a} \) is invariant from (53), if the frequency is constant the number of excitations is then also constant. Nonetheless, in the time dependent case, the number of excitations is inversely proportional to the time dependent frequency in correspondence with the intensity dependence obtained in the classical limit.

### 3.3 Minimum uncertainty states

The outstanding property of coherent states is that the motion of the center of mass of the wave packet corresponds to the classical mechanics variables (see for instance [60]) i.e.

\[
\langle \alpha | \hat{q} | \alpha \rangle = q_c, \quad \langle \alpha | \hat{p} | \alpha \rangle = p_c, \quad \langle \alpha | \hat{H} | \alpha \rangle = H_c
\]

(63)

where the sub-index \( c \) labels the classical variables. Coherent states are eigenstates of the annihilation operator \( \hat{b}|\alpha\rangle = \alpha|\alpha\rangle \). They are usually defined in the context of the time
independent QMHO. The Hamiltonian (36) with time independent unit frequency $\Omega = 1$ possesses annihilation and creation operators

$$\hat{b} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad \hat{b}^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}),$$

such that we can rewrite the Hamiltonian as (we set $\hbar = 1$)

$$\hat{H} = \omega_0 \left( \hat{b}^\dagger \hat{b} + \frac{1}{2} \right) = \omega_0 \left( \hat{n} + \frac{1}{2} \right).$$

(65)

Eigenstates for the Hamiltonian (65) are called Fock or number states $\hat{H}|n\rangle = \omega_0 \left( n + \frac{1}{2} \right) |n\rangle$. Fock states are orthonormal and form a complete basis. Unitary transformations then provide a useful tool to simplify the invariant and construct more general states, such as coherent or squeezed states of the QM-TDHO.

In the time dependent case, the quantum Ermakov-Lewis invariant (61) plays an analogous role to the Fock Hamiltonian (65) in the time independent case. Recall that a coherent state may be written in terms of the vacuum state according to (55). If we consider the initial state to be

$$|\psi(0)\rangle = \hat{T}^\dagger_{\rho} |\alpha\rangle = |\alpha\rangle_0$$

(66)

with $\alpha$ given in (55). The evolved state has the form

$$|\psi(t)\rangle = \hat{T}^\dagger_{\rho} |\alpha\rangle \exp \left( -i \int_0^t \omega(t') dt' \right) = |\alpha\rangle \exp \left( -i \int_0^t \omega(t') dt' \right)_t,$$

(67)

that is, coherent states keep their form through evolution. In the time dependent case, we define the operators

$$\hat{Q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{P} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger).$$

These operators are related to $\hat{q}$ and $\hat{p}$ by the transformations $\hat{q} = \hat{T}_{\rho} \hat{Q} \hat{T}_{\rho}^\dagger$ and $\hat{p} = \hat{T}_{\rho} \hat{P} \hat{T}_{\rho}^\dagger$. They obey the equations of motion [57] $\dot{\hat{Q}} = i\omega [\hat{I}_\rho, \hat{Q}] = \omega \hat{P}$ and $\dot{\hat{P}} = i\omega [\hat{I}_\rho, \hat{P}] = -\omega \hat{Q}$. The uncertainty relation for operators $\hat{Q}$ and $\hat{P}$ for the coherent state (55) is given by $\Delta \hat{Q} \Delta \hat{P} = \frac{1}{2}$, where $\Delta \hat{X} = \sqrt{\langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2}$ for $\hat{X} \rightarrow \hat{Q}, \hat{P}$. Time dependent coherent states are thus minimum uncertainty states (MUS), not for position and momentum but for the transformed position and momentum.

### 3.3.1 Step function solutions

Let us consider the Hamiltonian of the system to be given by equation (36) with $\Omega$ given by a step function that may be modeled by the hyperbolic tangent function stated in equation (22), where the frequency difference is $\Delta = \omega_2 - \omega_1$ and the initial and final frequencies are $\omega_0 \rightarrow \omega_1$ and $\omega_2$. The parameter $\epsilon$ describes the step function in the limit $\epsilon \rightarrow \infty$. The
Figure 2: $\rho(t)$ as a function of $t$ for $\omega_1 = 1$ and $\omega_2 = 2$ (solid line) and $\omega_2 = 3$ (dash line). $\epsilon = 20$ and $t_s = 2$. 
solution to the nonlinear amplitude equation is then given by (21). A plot of $\rho(t)$ is shown in figure 2 for an initial state with $\omega_1 = 1$ and a final state with either $\omega_2 = 2$ or $\omega_2 = 3$. Consider an initial coherent state (55) at time $t = 0$. From Fig. 2 we can see that $\hat{T}(0) = 1$ since $\dot{\rho} = 0$ and $\ln \rho = 0$. Therefore from (66), $|\psi(0)\rangle = \hat{T}_\rho^\dagger(0)|\alpha\rangle = |\alpha\rangle_0 = |\alpha\rangle$ and from (47) we obtain the evolved wave function

$$|\psi(t)\rangle = \exp \left[ -i \hat{T}_\rho \int_0^t \omega dt \right] \hat{T}_\rho^\dagger |\alpha\rangle = \hat{T}_\rho^\dagger |\alpha\rangle \exp \left( -i \int_0^t \omega dt \right).$$

(68)

Note that the coherent state in the above equation is given in the original Hilbert space, i.e. in terms of number states given in (65). From fig. 2 we can also see that for the maxima, $\dot{\rho}(t_{max}) = 0$ and $\ln \rho(t_{max}) = 0$, therefore $\hat{T}_\rho^\dagger(t_{max}) = 1$ and

$$|\psi(t_{max})\rangle = |\alpha\rangle \exp \left( -i \int_0^{t_{max}} \omega dt \right),$$

(69)

i.e. we recover the initial coherent state. However, for the minima, we have $\dot{\rho}(t_{min}) = 0$ and $\ln \rho(t_{min}) \neq 0$ and then we obtain

$$|\psi(t_{min})\rangle = \exp \left[ \frac{i \ln \rho(t_{min})}{2} (\hat{q}\hat{p} + \hat{p}\hat{q}) \right] |\alpha\rangle \exp \left( -i \int_0^{t_{min}} \omega dt \right),$$

(70)

that are the well-known squeezed (two-photon coherent) states [45]. Squeezed states, just as coherent states, are also MUS. However the uncertainties for $\hat{q}$ and $\hat{p}$ are $\Delta \hat{q} = 1/\sqrt{2\rho^{-1}(t_{min})}$ and $\Delta \hat{p} = 1/\sqrt{2\rho(t_{min})}$. For times in between neither coherent states nor standard squeezed states are obtained (in the initial Hilbert space), but the wave function

$$|\psi(t)\rangle = e^{-i\frac{\hat{p}^2}{2\rho}} |\alpha\rangle \exp \left( -i \int_0^t \omega(t') dt' \right) \ln \rho).$$

(71)

It should be stressed however, that in the instantaneous Hilbert space we will always have a coherent state (67).

The QM-TDHO with a step frequency function has been studied by Kiss et al. [34] and Agarwal and coworkers [35]. They have tackled the problem by dividing it in two regions for the two different frequencies of the step function and subsequently matching the solutions at the step. In contrast, the above procedure involves a continuous approach that allow us to obtain analytic solutions that do or do not exhibit squeezing depending on time. A quantum mechanical oscillator with a sudden mass change has been undertaken with an analogous procedure [32]. However, a Darboux transformation has been employed in order to avoid unwelcome divergences in the time dependent parameter as mentioned in section 2.2.3.

### 4 Wave propagation in deterministic inhomogeneous media

The previous time dependent oscillator problem is formally equivalent to the electromagnetic propagation of monochromatic waves in deterministic one dimensional inhomogeneous media as we shall readily show. Previous results may therefore be translated into
a nonlinear amplitude differential equation with an arbitrary refractive index variation. A hyperbolic tangent refractive index profile is one such example. The orthogonal functions invariant becomes a conserved quantity for the 3 + 1 dimensional scalar wave equation. The concept of orthogonal functions is extended to complementary fields that are always out of phase (subsection 4.2). The conserved density is not positive definite; However, its sign is merely a result of which of the two complementary fields is taken as reference as shown in 4.3. The density and its associated flow are evaluated using two different procedures; Namely, an extension of the one dimensional formalism or invoking the derivative of the reference field. The PDE in amplitude and phase variables permits the retrieval of the wave’s phase from non-interferometric squared amplitude measurements as described in 4.4. The nonlinear vector amplitude PDE is treated in the last subsection. The density and flow arising from the complementary fields formalism is compared with the Walther Marchand and Wolf radiometric theory.

4.1 One dimensional propagation

The propagation of plane electromagnetic waves in inhomogeneous media has been successfully described either when the permittivity variation takes place either in a much larger distance than the wavelength scale or in the abrupt case. In the former limit, the amplitude derivatives are neglected on a wavelength scale leading to the eikonal or ray equation [61]. In the latter, the usual procedure at a discrete boundary is to solve Maxwell’s equations in two homogeneous media with constant permittivities say, $\varepsilon_1$ and $\varepsilon_2$. The wave solutions for each region are then coupled at the interface by imposing the appropriate continuity conditions. Thin films and stratified media have been described by this latter procedure using a stack of constant permittivity media [62] and if necessary letting their thickness become very small [63].

The approach described here is to consider an inhomogeneous medium with an arbitrary space dependent permittivity. In order to simplify the problem, the description is restricted to one-dimensional propagation in a transparent medium whose permittivity gradient is orthogonal to the polarization. The resulting nonlinear equation derived for the field amplitude is recognized as the Ermakov - Pinney equation. Numerical solutions are presented for a refractive index that varies spatially as a hyperbolic tangent function. Recasting these results in terms of two counter propagating waves allow for a more useful description in terms of the reflectivity as a function of the abruptness of the interface.

The electric field equation arising from Maxwell’s equations for non magnetic inhomogeneous media without free charges is given by

$$\nabla^2 \vec{E} - \frac{\varepsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\nabla \left( \vec{E} \cdot \nabla \ln \varepsilon \right),$$  \hspace{1cm} (72)$$

where $c$ is the velocity of light in vacuum and $\varepsilon$ is the space dependent permittivity. If the permittivity only varies in the $z$ direction and the problem is restricted to plane waves at normal incidence to the constant permittivity planes, then $\frac{\partial \vec{E}}{\partial y} = 0$ and $\frac{\partial \vec{E}}{\partial x} = 0$. The
electric field equations for the \( x \) or \( y \) polarization in this case, are

\[
\frac{\partial^2 E_{x,y}}{\partial z^2} = \epsilon \frac{\epsilon^2}{c^2} \frac{\partial^2 E_{x,y}}{\partial t^2}.
\] (73)

Whereas the field in the \( z \) direction obeys the equation

\[
\frac{\partial^2 E_z}{\partial z^2} + \frac{\partial}{\partial z} \left( E_z \frac{\partial \ln \varepsilon}{\partial z} \right) = \epsilon \frac{\epsilon^2}{c^2} \frac{\partial^2 E_z}{\partial t^2}.
\]

From the first of Maxwell’s equations \( \nabla \cdot (\varepsilon \vec{E}) = 0 \), \( \frac{\partial E_z}{\partial z} = -E_z \frac{\partial \ln \varepsilon}{\partial z} \) thus, the second order time derivative of \( E_z \) is equal to zero. The field in this direction is not a propagating field and is therefore not involved in the transmission of the electromagnetic wave. The equation for the \( \hat{i} \) or \( \hat{j} \) electric field component with a monochromatic temporal dependence is then proposed to be

\[ E_{x,y}(z,t) = E_{x,y}(z) e^{-i\omega t}. \] (74)

The spatially dependent electric field function is equivalent for either direction and conforms with the equation

\[
\frac{\partial^2 E(z)}{\partial z^2} = -\varepsilon(z) k_0^2 E(z),
\] (75)

where we have substituted the wave vector magnitude in vacuum \( k_0 = \omega/c \). This equation is formally identical to the TDHO equation and therefore the results obtained in section 2 may be readily employed. The temporal variable in dynamical systems is substituted here by the direction of propagation (say the spatial coordinate \( z \)). The electric field in terms of the amplitude is then an exponential integral function

\[
E = A \exp \left[ i \left( \int \frac{W}{A^2} \, dz - \omega t \right) \right],
\]

where the spatially dependent field amplitude fulfills the nonlinear amplitude equation

\[
\frac{\partial^2 A}{\partial z^2} - \frac{W^2}{A^3} = -\varepsilon k_0^2 A.
\] (76)

This equation is formally equivalent to the time dependent harmonic oscillator equation (6) with the time variable replaced by the spatial coordinate. From the nonlinear superposition principle stated in subsection 2.3, the general amplitude solution (15) may arise from two opposite phase contributions

\[
E = Ae^{i(\gamma(z) - \omega t)} = ae^{i\delta - i\omega t} + be^{-i\delta - i\omega t}
\] (77)

with

\[
A = \sqrt{a^2 + b^2 + 2ab \cos (2\delta)}.
\] (78)

The two exponentials in (77) are of course interpreted as waves propagating in opposite directions. The total amplitude \( A \) arises then from the interference of the opposite waves.
The standing waves interference pattern is usually stated in terms of the intensity $I = A^2$, so that the square of (76) gives the customary expression

$$I = I_a + I_b + 2 \sqrt{I_a I_b} \cos (2s).$$

In subsection 2.5 we discussed the indeterminacy in the amplitude and phase variables and the identical trajectories thereby produced. The different amplitude and phase solutions in the propagation-less case were then indistinguishable. However, in the present state of affairs, propagation actually establishes two distinct situations.

The orthogonal functions invariant from (17) or (30) for a constant parameter is

$$W = AB \dot{\phi} = (a^2 - b^2) \dot{s}.$$ 

The condition $B = A$ implies $b = 0$ and in this propagating context it evinces that there is solely a single propagating wave. $B \neq A$ necessarily entails the existence of measurable counter propagating waves. Furthermore if $B = 0$ then $a = b$ and the counter propagating waves have equal amplitude. The fact that the invariant is zero in this latter case indicates that this quantity is a measure of the field flow since two opposite but equal flows produce a zero net flow. This assertion is further developed in subsection 4.2 where the scalar wave equation is appraised.

### 4.1.1 Numerical solutions

Propagation of electromagnetic waves in media with refractive index changes in the order of the wavelength cannot be dealt with the slowly varying permittivity approximation nor the abrupt interface scheme. It is then necessary to evaluate the nonlinear amplitude equation without approximations. To this end, consider a hyperbolic tangent refractive index spatial variation with an arbitrary slope analogous to the time dependent function used in (22). Allow for the refractive index to be written as

$$n(z, D) = n_{\text{min}} + \left( n_{\text{max}} - n_{\text{min}} \right) \left( 1 + \tanh \left[ \epsilon (D) z \right] \right), \quad (79)$$

where $n_{\text{max}}$ and $n_{\text{min}}$ are the maximum and minimum refractive indices obtained in the limit where the refractive index is constant. The maximum slope of this function is exhibited at $z = 0$. In optical problems it is customary to describe the refractive index change in terms of the typical distance where the variation takes place rather than its slope. For this reason, a thickness parameter $D$, defined by

$$\epsilon = \frac{2}{D} \arctan \left( \frac{9}{10} \right) \quad (80)$$

is introduced. It corresponds to the thickness over which the refractive index varies within 90% of its initial and final values. Let the invariant be equal to $W = k_0 A_0^2$ at a given position $z_0$. Normalize to wavelength units so that $k_0 = 2\pi/\lambda = 2\pi$ and introduce the
dimensionless amplitude $A_d = A/A_0$. The Ermakov differential equation to be numerically solved is then

$$
(2\pi)^{-2} \frac{\partial^2 A_d}{\partial z^2} - \frac{1}{A_d^3} = -[n(z,D)]^2 A_d.
$$

The important issue here is to establish the adequate conditions in order to solve this equation. At first sight, the reasonable choice would be to consider the incident field amplitude and its derivative at a given plane. However, such a proposal is inadequate since at any plane where the incident wave exists there is also a contribution from the reflected wave, which is so far unknown. This assertion is true even far away from the region where the refractive index varies considerably since we are dealing with infinite wave trains. The appropriate alternative is to establish the conditions for the refracted wave, which in this one-dimensional case is the transmitted wave. The assumption then required is that far from the interface region the transmitted wave is constant and in this region there is no reflected wave. Thus the problem is like working backwards in time and obtaining the incident and reflected waves from the transmitted wave.

Consider that the incident wave travels from $z \to -\infty$ towards the positive $z$ direction and the refractive index change takes place around $z = 0$. The initial (final) conditions for the above nonlinear amplitude differential equation are then

$$
\left. \frac{\partial A_d}{\partial z} \right|_{z=z_1} = 0, \quad A_d(z_1) = (n_{\text{max}})^{-1/2}, \quad z_1 \gg 0,
$$

where $A_d(z_1)$ is the dimensionless transmitted amplitude far from the interface in the positive semi-space. Various solutions of equation (81), together with the conditions imposed by (82) are plotted in figure 3. In these graphs, oscillations reveal the existence of counter propagating waves according to the results obtained in the previous section. The larger the oscillation, the larger is the reflected wave amplitude. It may be seen that if the refractive index change takes place in the order of one or more wavelengths, the reflectivity is almost null. When this transition occurs in less than a wavelength there is a considerable increase in the reflectivity. Finally, when the transition thickness $D$ reaches a fiftieth of a wavelength or so, the reflectivity approaches a maximum, which barely increases for more steep index changes approaching the abrupt limit [64].

It is important to remark that if we attempt to numerically solve the simpler electric field linear differential equation (75) it is not clear which conditions should be imposed on the field and its first derivative. However, the amplitude and phase representation separates the contributions of the space and time oscillating terms. The resulting nonlinear amplitude equation albeit more complex than the field linear equation, is better suited in order to establish the initial conditions as shown in the above example. Furthermore, a smooth or

1There is an errata in ref. [64], eq. (21), where the factor on the left should be $(2\pi)^{-2}$ instead of $(2\pi)^2$. The correct equation (81) is presented here.

2This statement should not be confused with a slowly varying approximation (SVA) where higher derivatives of sluggish oscillations are dismissed. The amplitude, in this case, may still exhibit rapid spatial oscillations as shown in figure (3).
Figure 3: Numerical results showing the dimensionless amplitude of the electric field \( A_d \) as a function of distance for different refractive index variation thickness’s \( D ( = \lambda, \lambda/2, \lambda/4, \lambda/10, \lambda/50, \lambda/500) \). Pairs of curves are presented depicting the field amplitude solutions \( (A_d \text{ vs } z) \) with their corresponding refractive index variation \( (n \text{ vs } z) \). Notice that this amplitude does not distinguish between incident, reflected and transmitted waves but represents the actual overall field amplitude at any point.

oscillating amplitude behaviour may be readily identified with a running wave or counter propagating waves respectively.

The amplitudes of the incident and reflected waves \( a (z \to -\infty) \) and \( b (z \to -\infty) \) can be calculated from the numerical values of the maxima and minima in the oscillating region, far away from the interface

\[
a_0 = \frac{1}{2} (A_d \text{ max} + A_d \text{ min}) , \quad b_0 = \frac{1}{2} (A_d \text{ max} - A_d \text{ min}) .
\]

The reflectivity, defined as the ratio of the reflected squared amplitude over the incident squared amplitude, can be obtained from the numerical solutions for various refractive indices with different spatial variations. Figure 4 depicts a plot of the reflectivity versus the 90% refractive index variation thickness \( D_\lambda \) (wavelength normalized). The data were taken from 100 different numerical solutions of the nonlinear amplitude equation (81) for different values of \( D_\lambda \). The abrupt interface is obtained as \( D_\lambda \) tends to zero, the reflectivity then
tends to the expected value of \( \frac{(n_{\text{max}} - n_{\text{min}})^2}{(n_{\text{max}} + n_{\text{min}})^2} \) obtained for normal incidence from the Fresnel equations. The reflectivity remains almost constant for thicknesses \( D_\lambda \) smaller than 0.05. For larger thickness it decreases monotonically and the reflectivity becomes negligible for refractive index variation thicknesses over one wavelength.

4.2 Scalar wave equation

The orthogonal functions derivation can be extended to the 3+1 dimensional scalar wave equation as we shall presently demonstrate. For systems with one degree of freedom, the assessed quantity becomes the previously known invariant. In the case of a complex perturbation, the conserved quantity and its flow are shown to be equivalent to their counterparts in the previous real orthogonal functions derivation. These expressions have been studied before in the context of the Klein Gordon equation [65]. Due to the lack of a positive definite density and thus the impossibility to represent a probability density, this continuity equation has often been dismissed. Nonetheless, this charge like density will be shown to emanate from the presence of two out of phase fields. This interpretation is plausible for wave phenomena where the disturbance comes from the imbalance between two forms.
of energy \[4\]. The complementary field or linearly independent function is then evaluated from a given solution using two different procedures. The density and flow obtained from these results are compared with the positive definite density conservation equation usually invoked to evaluate the energy content of a wave fulfilling a scalar second order differential equation.

### 4.2.1 Complementary functions procedure

In order to obtain a continuity equation of the form \(\nabla \cdot \mathbf{J} + (\partial/\partial t) \rho = 0\), consider the following procedure. The starting point is the scalar wave equation

\[
\nabla^2 \psi (r, t) - \frac{1}{v^2} \frac{\partial^2 \psi (r, t)}{\partial t^2} = 0, \tag{83}
\]

where the scalar \(\psi\) represents the disturbance and \(v\) is the velocity of propagation. Allow for two real linearly independent solutions of the wave equation to be \(\psi_1 (r, t)\) and \(\psi_2 (r, t)\); (the notation that exhibits the space and time dependence \((r, t)\) is dropped in the derivation and shown explicitly only when needed). Perform the product of \(\psi_2\) times the wave equation for \(\psi_1\):

\[
\psi_2 \left(\nabla^2 \psi_1 - \frac{1}{v^2} \frac{\partial^2 \psi_1}{\partial t^2}\right) = 0.
\]

Calculate the product inverting the solutions and evaluate their difference:

\[
(\psi_2 \nabla^2 \psi_1 - \psi_1 \nabla^2 \psi_2) + \frac{1}{v^2} \left(\psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t}\right) = 0.
\]

This equation, with the aid of Green’s theorem, may be written as

\[
\nabla \cdot (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2) + \frac{1}{v^2} \frac{\partial}{\partial t} \left(\psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t}\right) = 0, \tag{84}
\]

where the assessed quantity \(\psi_\rho\) and its corresponding flux \(\triangleright \psi_\rho\) are defined as

\[
\psi_\rho \equiv \left(\psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t}\right), \quad \triangleright \psi_\rho \equiv (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2). \tag{85}
\]

Provided that the medium does not exhibit dispersion but is allowed to be inhomogeneous, the above expression may be written as a continuity equation

\[
\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t} \mathcal{E} = 0, \tag{86}
\]

where the density of the conserved quantity and its associated flux are

\[
\mathcal{E} = \frac{1}{v^2 (r)} \left(\psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t}\right), \quad \mathbf{S} = (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2). \tag{87}
\]
These definitions may be scaled by an arbitrary constant, in particular, a factor of $1/2$ will be included when these quantities are compared with the usual energy and momentum flow variables. If the medium were homogeneous but with a time dependent velocity, a continuity equation may also be obtained

$$\nabla \cdot \left[ v^2(t) \left( \triangledown \psi_\rho \right) \right] + \frac{\partial}{\partial t} \left( \psi_\rho \right) = 0, \quad (88)$$

provided that the density and flux are now defined by $\mathcal{E}_{\text{disp}} = \psi_\rho$ and $\mathbf{S}_{\text{disp}} = v^2(t) \left( \triangledown \psi_\rho \right)$ respectively. In a homogeneous dispersion-less medium either definition may, of course, be used. The quantities in (85) may be written as

$$\psi_\rho = \psi_1^2 \frac{\partial}{\partial t} \left( \frac{\psi_2}{\psi_1} \right) = -\psi_2^2 \frac{\partial}{\partial t} \left( \frac{\psi_1}{\psi_2} \right), \quad \triangledown \psi_\rho = -\psi_2^2 \nabla \left( \frac{\psi_2}{\psi_1} \right);$$

so that any one solution may be readily expressed in terms of the other function and the density, i.e. $\psi_2 = \psi_1 \int \left( \psi_\rho / \psi_1^2 \right) \, dt$. Linear independence of the solutions in the temporal variable is insured if the scalar field density does not vanish, $\psi_\rho \neq 0$. Whereas linear independence in the spatial variables is obtained if the vector field $\triangledown \psi_\rho \neq 0$. The wave equation insures that if the two fields are linearly independent in the temporal variable then they are also independent in the spatial variables. The only trivial exception being if the velocity of propagation is zero. Hereafter, we shall refer to these linearly independent solutions in the temporal and spatial variables as *complementary fields*.

Orthogonality, in the analytic functions sense, of the linearly independent solutions $\psi_1(r, t)$ and $\psi_2(r, t)$ over the interval $[a, b]$ with respect to the weight function $w$ may be achieved through Schmidt’s method. Bi-orthonormal systems of this sort have been used to identify the radiative and non-radiative parts of a wave field [66]. If the density is evaluated using the linearly independent solution $\psi_1$ albeit not necessarily orthogonal to $\psi_2$, then

$$\psi_\rho = -\psi_2^2 \frac{\partial}{\partial t} \left( \frac{\psi_1}{\psi_2} \right) = -\psi_2^2 \frac{\partial}{\partial t} \left( \frac{\psi_{1\perp} + \lambda_{12} \psi_2}{\psi_2} \right) = -\psi_2^2 \frac{\partial}{\partial t} \left( \frac{\psi_{1\perp}}{\psi_2} \right).$$

Therefore, the contribution of the non-orthogonal component is zero and thus the non-vanishing contribution to the density comes from the orthogonal fields solutions. An analogous procedure may be performed in the spatial domain in order to derive the orthogonal spatial field. The non-zero contribution to the flow is again obtained from the spatially orthogonal field solution. The complementary field function is, of course, not unique since any linearly dependent function may be added without altering the density function $\psi_\rho$. In addition, as it is well known, a continuity equation admits a density that is defined up to a time independent scalar $\mathcal{E}' = \mathcal{E} + G(r)$ and a flow with an arbitrary divergence free field $\mathbf{S}' = \mathbf{S} + \triangledown \times \mathbf{G}$. Furthermore, for any twice differentiable vector field $\mathbf{G}$, a modified density $\mathcal{E}' = \mathcal{E} + \triangledown \cdot \mathbf{G}$ and flux $\mathbf{S}' = \mathbf{S} - \partial \mathbf{G} / \partial t$ are also admissible [67].
4.2.2 Systems with restricted degrees of freedom

spatially harmonic field

In the particular case where the field functions are harmonic in the spatial domain

\[ \nabla^2 \psi(r, t) = -k^2(t) \psi(r, t), \]  

(89)

where the wave vector magnitude \( k^2(t) \) is spatially constant but has an arbitrary time dependence. The complementary functions procedure applied to (89) yields \( \nabla \cdot (\nabla \psi) = 0 \). The continuity equation (84) then leads to an invariant

\[ \psi_r \rightarrow W = \psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t}. \]  

(90)

Substitution of the spatial harmonic dependence (89) in the wave equation (83) gives

\[ k^2(t) \psi(r, t) + \frac{1}{v^2(t)} \frac{\partial^2 \psi(r, t)}{\partial t^2} = 0. \]

The standard separation of variables \( \psi(r, t) = \psi_{sp}(r) \psi_t(t) \) then yields decoupled differential equations for the spatial and temporal behaviour. The temporal equation is equal to the time dependent oscillator equation with time dependent parameter given by \( \Omega^2(t) = k^2(t) v^2(t) \). Therefore, the harmonic spatial field restriction, as it is well-known, describes a continuum problem that fulfills a temporal differential equation identical to that obtained from the discrete non-propagating problem of a single particle in a time dependent potential. The density obtained in the different formalisms under this restriction will be discussed in the following sections.

Given one field solution, say \( \psi_{t1}(t) \), the complementary field may be readily obtained from the invariant relationship (90) in this one dimensional case

\[ \psi_{t2}(t) = \psi_{t1}(t) \int \frac{W}{\psi_{t1}^2(t)} dt. \]

This is the common way to obtain, with the aid of the Wronskian \( W \), a linearly independent solution in ordinary second order differential equations [68, p. 402]. In terms of real amplitude \( a \) and phase \( s \) variables, the one dimensional exact invariant (7) is \( W = a^2 \frac{ds}{dt} \). Given a solution of the form \( \psi_{1t}(t) = a(t) \cos[s(t)] \), the linearly independent solution is

\[ \psi_{2t}(t) = a(t) \cos[s(t)] \int \frac{a^2(t) ds/dt}{a^2(t) \cos^2[s(t)]} dt = a(t) \sin[s(t)]. \]  

(91)

monochromatic field

A field with harmonic time dependence transforms the wave equation into the time independent diffusion equation

\[ \nabla^2 \psi(r) = -\frac{\omega_0^2}{v^2(r)} \psi(r), \]
where the medium inhomogeneity $\kappa^2 (r) = \omega_0^2 / v^2 (r)$ is now constant in time but has an arbitrary spatial dependence. The solution in amplitude and phase variables is given by

$$\psi_1 (r, t) = a(r) \cos (\phi(r) - \omega_0 t + \varphi_0), \quad (92)$$

where $\varphi_0$ is a constant phase. Since the continuity equation then reads as $\nabla \cdot (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2) = 0$, the obtention of the linearly independent solution is not straightforward from this result. Nonetheless, the harmonic time dependence yields a time independent density and thus (90) is fulfilled. The linearly independent solution may then be proposed to be

$$\psi_2 (r, t) = a(r) \sin (\phi(r) - \omega_0 t + \varphi_0). \quad (93)$$

The density and flow are then

$$\psi_\rho = -a^2 (r) \omega_0, \quad \nabla \cdot \psi_\rho = -a^2 (r) \nabla \phi (r). \quad (94)$$

According to this expression, the flow is finite in any direction where the wave vector has a non vanishing projection. Notice that the general solution (92) and the linearly independent solution (93) are being used to obtain the density and its corresponding flow. The usual sequence in ordinary differential equations is the opposite where the general solution is obtained from a particular solution with the help of a constant density. In an unrestricted system, the density becomes in general spatially and time dependent. The problem is then to find an independent solution in order to evaluate the density. The continuity equation for a monochromatic field in amplitude and phase variables is from (86) and (94)

$$\nabla \cdot S = \nabla \cdot \left[ a^2 (r) \nabla \phi (r) \right] = 0$$

since the density is time independent. Furthermore, the one dimensional spatial restriction, say in the $z$ direction, reduces to an invariant of the form

$$Q = a^2 (z) \frac{\partial \phi (z)}{\partial z}. \quad (95)$$

This expression is equal to the Wronskian (7) with the substitution $z \rightarrow t$. This invariant allows for the decoupling of the amplitude and phase equations in the one dimensional case.

### 4.2.3 Complex disturbance

In the complex case, just as in the real disturbance derivation, there is no contribution to the density or flow for complementary field terms that are linearly dependent with the reference field [31]. It is therefore sufficient, without loss of generality, to introduce linearly independent complex solutions; Allow for the complex complementary fields to be

$$\tilde{\psi} = b_{1r} \psi_1 + b_{2i} \psi_2 i, \quad \tilde{\psi}^{(comp)} = b_{1r} \psi_1 - b_{2i} \psi_2 i. \quad (96)$$
These two solutions are linearly independent provided that \( b_{1r} \) and \( b_{2i} \) are non zero; that is, the solutions must neither be purely real nor purely imaginary. To wit, the density

\[
\tilde{\psi}_\rho = 2ib_{1r}b_{2i} \left( \frac{\psi_1}{\partial t} \frac{\partial \psi_2}{\partial t} - \frac{\psi_2}{\partial t} \frac{\partial \psi_1}{\partial t} \right)
\]

is finite since \( \psi_1, \psi_2 \) are linearly independent solutions. An alternative procedure that may be employed to obtain this result is to recreate the complementary functions procedure using the pair \( \tilde{\psi}^* \) and \( \tilde{\psi} \) rather than the linearly independent real solutions \( \psi_1, \psi_2 \). The continuity equation then reads

\[
\nabla \cdot \left( \tilde{\psi} \nabla \tilde{\psi}^* - \tilde{\psi}^* \nabla \tilde{\psi} \right) + \frac{1}{\psi_1^2} \frac{\partial}{\partial t} \left( \tilde{\psi} \frac{\partial \tilde{\psi}^*}{\partial t} - \tilde{\psi}^* \frac{\partial \tilde{\psi}}{\partial t} \right) = 0,
\]

where the assessed quantity \( \psi_\rho \) and its corresponding flux \( \triangleright \psi_\rho \) are now defined by

\[
\psi_\rho \equiv \frac{1}{2i} \left( \tilde{\psi} \frac{\partial \tilde{\psi}}{\partial t} - \tilde{\psi}^* \frac{\partial \tilde{\psi}^*}{\partial t} \right), \quad \triangleright \psi_\rho \equiv \frac{1}{2i} \left( \tilde{\psi} \nabla \tilde{\psi}^* - \tilde{\psi}^* \nabla \tilde{\psi} \right).
\] (97)

The factors \( 1/2i \) have been introduced in order to obtain real expressions for these quantities. The assessed quantity and its flow are equal either for the real linearly independent solutions (85) or the complex solution

\[
\tilde{\psi} = \psi_1 + \psi_2 i,
\] (98)

together with its concomitant definitions of density and flow (97) as may be seen from direct substitution. The constants \( b_{1r}, b_{2i} \) have been set to one since the solutions can always be rescaled. The complementary field in the complex formalism is the complex conjugate of the original field \( \tilde{\psi}_{comp} = \tilde{\psi}^* \). The expression for the density in terms of complex conjugate fields (97) is encountered when dealing with the Klein Gordon Schrödinger equation [69, p. 474]. Nonetheless, it is commonly dismissed because it is not positive definite. So far, according with the present results, it is clear that this density corresponds to a quantity assessed between two complementary fields. The relative phase between these fields, as we shall discuss in the next section, defines the sign of this quantity.

In harmonic time dependent phenomena evaluation of a product involving the function and its conjugate is often used as a mathematical technique in order to perform an average [70]. The above derivation with a complex field may be misleading because it may be thought that the complex conjugate expressions involve some sort of averaging. However, in the complementary functions procedure with real linearly independent solutions no average was performed at all. Since the evaluation of the density and flow (97) with the complex function (98) is entirely equivalent to the complementary real functions method, these expressions do not entail any sort of averaging.
4.3 Complementary field evaluation

In the case of a disturbance with arbitrary spatial and time dependence, the independent solution cannot be readily obtained as in second order differential equations in one variable. In this general case, given one solution, there are different possibilities that may be pursued in order to obtain a linearly independent solution. Two such possibilities are explored hereafter. In the first subsection, a generalization of the amplitude and phase representation is invoked in order to obtain a linearly independent solution. This proposal generates a flow that is always orthogonal, in the vector sense, to the wave front even in the time dependent case. In the second subsection, the time derivative of the wave equation is used to produce the second solution. Under certain circumstances the resulting density becomes positive definite. An asset of this approach is that the one dimensional restriction yields a density that is equal to the energy of the discrete system.

4.3.1 Orthogonal trajectories

The general solution of the wave equation in terms of real amplitude and phase variables is

\[ \psi_1 (r, t) = a(r, t) \cos(s(r, t) + \varphi_0), \] (99)

where \( \varphi_0 \) is a constant phase. Following a generalization of the previous results \( i.e. \) (91) and (93), the complementary field is proposed to be

\[ \psi_2 (r, t) = \mp b_\perp a(r, t) \sin(s(r, t) + \varphi_0). \] (100)

The complementary field is then a function that is \( 90\frac{1}{2} \) out of phase with respect to the original field. The complex solution (98) in terms of real amplitude and phase variables from (99) and (100) is \( \tilde{\psi} = a(r, t) \cos(s(r, t)) \mp b_\perp a(r, t) \sin(s(r, t)) i \) and its polar representation is

\[ \tilde{\psi} = a(r, t) \left[ 1 + (b_\perp^2 - 1) \sin^2(s(r, t)) \right]^{\frac{1}{2}} \exp \{ \mp i \arctan b_\perp \tan(s(r, t)) \}, \] (101)

where the constant phase has been set to zero. If the complementary field constant coefficient is normalized, \( b_\perp = 1 \), the polar expression for the complex field is simply

\[ \tilde{\psi} = a(r, t) e^{\mp i(s(r, t) + \varphi_0)}. \]

The terms involving temporal derivatives of the conserved quantity (85) are

\[ \psi_1 \frac{\partial \psi_2 (r, t)}{\partial t} = \left[ \mp b_\perp a^2 \frac{\partial s}{\partial t} \cos^2(s) \pm b_\perp a \frac{\partial a}{\partial t} \sin(s) \cos(s) \right] \]

and

\[ \psi_2 \frac{\partial \psi_1 (r, t)}{\partial t} = \left[ \mp b_\perp a^2 \frac{\partial s}{\partial t} \sin^2(s) \pm b_\perp a \frac{\partial a}{\partial t} \sin(s) \cos(s) \right]. \]
The Nonlinear Amplitude Equation in Harmonic Phenomena

Figure 5: Phase difference between the two complementary fields. If the reference field \( \psi_1 \) is represented by the solid line, the complementary field \( \psi_2 \) (dotted line) lags by \( 90i\frac{\pi}{2} \); The density \( \psi \rho \), defined in (85) or in polar variables by (102), is then negative. If, on the contrary, the reference field \( \psi_1 \) is represented by the dotted line, then the complementary field \( \psi_2 \) (solid line) leads the reference field by \( 90i\frac{\pi}{2} \); The density is then positive.

so that their difference yield the density

\[
\psi^{(\perp)}_{\rho} = \pm b_{\perp} a^2 (r, t) \frac{\partial s (r, t)}{\partial t}.
\]  

If there is no complementary field, i.e. \( b_{\perp} = 0 \), the density is obviously zero. It is therefore crucial to have a finite complementary field in order to obtain a non trivial continuity equation. On the other hand, whether this quantity is positive or negative depends on whether the complementary field leads or lags the reference field (or for that matter, which field is taken as the reference [71]). The phase difference between the two complementary fields for constant amplitude is depicted in figure 5.

The main drawback of a density, which is not positive definite is that it may be inadequate to represent some variables such as the probability density in certain quantum mechanical problems or the energy density in classical waves. However, the present interpretation shows that the sign of the density relies on whether the complementary field leads or lags the reference field and as such, it may well be an appropriate variable for physical quantities where the reference of the equilibrium value can be arbitrarily set.

The terms with spatial derivatives in the complementary fields flow (85) follow an analogous derivation so that

\[
S_{\perp} = \nabla \cdot \psi^{(\perp)}_{\rho} = \pm b_{\perp} a^2 (r, t) \nabla s (r, t).
\]  

The spatially constant phase surfaces define the wave front. Since the flow defined above is zero for a spatially constant phase, the flow \( S_{\perp} \) is then perpendicular, or orthogonal in the vector sense, to the wave front even in the time dependent case. To wit, orthogonal trajectories are insured since from direct evaluation of the above expression \( S_{\perp} \cdot (\nabla \times S_{\perp}) = 0 \)
For a monochromatic wave, the density (102) and flow (103) are time independent although no averaging process has taken place in the derivation. For a plane wave, the phase spatial dependence is \( \phi (r) = k \cdot r \), where the wave vector \( k \) is constant. The assessed quantities are then also spatially constant

\[
\psi^{(\perp)}_\rho = b_\perp a_0^2 \omega_0, \quad S_\perp = b_\perp a_0^2 k; \tag{104}
\]

where the upper sign of the expressions has been taken. Therefore, plane wave propagation yields a constant density and flow in this formalism even without performing any averaging. Regarding the parity of these quantities, the density \( \psi^{(\perp)}_\rho \) is an odd function of time provided that the constant \( b_\perp \) remains invariant under time reversal. This result is not surprising since the field that lags by 90\(^\circ\) becomes a leading field by 90\(^\circ\) under the time transformation.

Under space inversion, \( \psi^{(\perp)}_\rho \) remains unaltered. On the other hand, the flow \( S_\perp \) is invariant under time reversal and odd under space inversion.

Introducing the complex disturbance (101) with \( b_\perp = 1 \) in the real part of the scalar wave equation yields

\[
\nabla^2 a - (\nabla s \cdot \nabla s) a - \frac{1}{v^2} \left[ \frac{\partial^2 a}{\partial t^2} - \left( \frac{\partial s}{\partial t} \right)^2 a \right] = 0 \tag{105}
\]

and the imaginary part gives

\[
a \nabla^2 s + 2 (\nabla s \cdot \nabla) a - \frac{1}{v^2} \left[ a \frac{\partial^2 s}{\partial t^2} + 2 \frac{\partial s}{\partial t} \frac{\partial a}{\partial t} \right] = 0.
\]

An amplitude \( a \) and frequency \( \omega = \dot{s} \) time independent version of these results is often used in optical scalar theory [73]. The latter equation, provided that the amplitude is finite, may be written as

\[
\nabla \cdot (a^2 \nabla s) - \frac{1}{v^2} \left[ \frac{\partial}{\partial t} \left( a^2 \frac{\partial s}{\partial t} \right) \right] = 0. \tag{106}
\]

However, this equation is precisely the conservation equation previously derived in (86) together with (102) and (103). Therefore, the continuity equation arising from the complementary orthogonal fields is also obtained when a complex disturbance of the form (101) is introduced in the wave equation. The amplitude and phase variables appear in both (105) and (106) partial differential equations. These equations cannot be decoupled unless further restrictions are imposed. The author is not aware of published work that states the necessary conditions that permit decoupling. In subsection 4.5, a sufficient condition when the amplitude is wave vector independent, is described.

### 4.3.2 Derivative field

A second possibility in order to obtain an independent solution is the following: In a dispersion-less medium, the time derivative of the wave equation is

\[
\nabla^2 \frac{\partial \psi}{\partial t} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \left( \frac{\partial \psi}{\partial t} \right) = 0.
\]
Since the function $\partial \psi / \partial t$ also satisfies a wave equation, a linearly independent solution may be obtained from the identification $\psi_1 \rightarrow \partial \psi / \partial t$ and $\psi_2 \rightarrow \psi$. The complementary fields density and flow from (85) are then

$$\psi^{(DF)}_\rho = \left( \frac{\partial \psi}{\partial t} \right)^2 - \psi \frac{\partial^2 \psi}{\partial t^2}, \quad \triangleright \psi^{(DF)}_\rho = \psi \nabla \left( \frac{\partial \psi}{\partial t} \right) - \frac{\partial \psi}{\partial t} \nabla \psi. \quad (107)$$

These expressions may be economically written as

$$\psi^{(DF)}_\rho = -\psi^2 \frac{\partial^2 \ln \psi}{\partial t^2}, \quad \triangleright \psi^{(DF)}_\rho = \psi^2 \nabla \left( \frac{\partial \ln \psi}{\partial t} \right). \quad (108)$$

From these results it follows that the density $\psi^{(DF)}_\rho$ remains invariant under time reversal and space inversion. On the other hand, the flow $\triangleright \psi^{(DF)}_\rho$ is an odd function under time reversal and space inversion. In order to describe the density and flow in amplitude and phase variables, let the solution be written as

$$\psi(r, t) = \psi_2(r, t) = a(r, t) \sin(s(r, t)).$$

The density is thus

$$\psi^{(DF)}_\rho = a^2 \left[ \left( \frac{\partial s}{\partial t} \right)^2 - \frac{\partial^2 \ln a}{\partial t^2} \sin^2 s - \frac{1}{2} \frac{\partial^2 s}{\partial t^2} \sin(2s) \right]. \quad (109)$$

and the flow is

$$\triangleright \psi^{(DF)}_\rho = a^2 \left[ -\nabla s \frac{\partial s}{\partial t} + \nabla \left( \frac{\partial \ln a}{\partial t} \right) \sin^2 s + \frac{1}{2} \nabla \left( \frac{\partial s}{\partial t} \right) \sin(2s) \right]. \quad (109)$$

The complex representation of these quantities are obtained from (97) together with

$$\tilde{\psi} = \frac{\partial \psi}{\partial t} + \psi i.$$

The density $\psi_\rho$ is not positive definite as has been mentioned before. However, if the field is harmonic in time, the density that arises from a complementary derivative field (107) becomes positive definite since

$$\psi^{(DF)}_\rho \text{ (harmonic)} = \left( \frac{\partial \psi}{\partial t} \right)^2 + \omega_0^2 \psi^2.$$

Therefore, if the field is decomposed in Fourier components, the contribution of each monochromatic component has a positive definite density in this scheme. In the monochromatic case, the density (and flow) defined from the orthogonal trajectories (102) or the derivative function (108) methods is the same $\psi^{(DF)}_\rho = \psi^{(\perp)}_\rho$ provided that $b_\perp = \omega_0$. In

\(^3\)the super-index (DF) in the density or the flow stands for Derivative Field.
either scheme these quantities are constant in time without having performed any average. The reason lies in the fact that, in both cases, two out of phase fields are being invoked. In order to elucidate this point recall, for example, the time independent one dimensional harmonic oscillator illustrated by a simple pendulum or a spring. The total time independent energy arises from two time dependent out of phase functions, namely, the kinetic and potential energy. This result is obtained in the present formalism from the derivative field density (107) by letting \( \psi \) represent the displacement of the particle; the complementary field \( \psi_1 = d\psi/dt \) then stands for the velocity of the particle. Thus the invariant density \( \psi^{(DF)}_\rho = \dot{\psi}^2 + (\kappa/m) \psi^2 \) scaled by a factor of \( m/2 \) is, in this example, equal to the total energy of the system. In contrast, the energy density usually associated with a scalar wave field (123) does not yield the energy of the discrete harmonic oscillator system in the monochromatic one dimensional limit. A second example of a positive definite density is the propagation of a Gaussian pulse [31].

4.4 Phase retrieval from intensity measurements

Different types of interferometric schemes have been the major technique used to obtain phase information from a wide variety of wave phenomena. Diffraction limited resolution in optical instruments is attainable with this approach [74]. However, the experimental setup usually requires stable well aligned systems together with an adequate coherence of the source. Non interferometric optical testing is usually performed in the framework of geometrical optics through the tracing of pencils of rays. The wave front or isophase curves are then reconstructed from the transverse aberration data [75, p.614]. However, the resolution obtained with these techniques is limited by the eikonal approximation [76], [77]. Furthermore, image reconstruction of phase objects is in most cases impractical using these procedures.

An alternative proposal developed in the last two decades has been to measure the irradiance in two planes orthogonal to the propagation direction in order to retrieve the phase [78]. The deterministic approach to phase retrieval from intensity records is based on the irradiance transport equation. This result was first derived using a parabolic equation as a starting point [79]. The retrieval of the phase from intensity measurements has received much interest in different fields of science ranging from wavefront sensing in optical testing [80] and adaptive optics [81] to visualization of phase objects in different regions of the electromagnetic spectrum [82].

Consider the propagation of an electromagnetic wave in an inhomogeneous isotropic linear medium. The electric field may be written in the polar representation with a complex vector amplitude \( \tilde{a} (r, t) \) and a real scalar phase \( \phi (r, t) \) as \( \mathbf{E}_g (r, t) = \tilde{a} (r, t) \exp [i \phi (r, t)] \). In the Fourier domain, the field is expressed in terms of infinite wave trains with fixed frequency \( \omega \), where each component satisfies Maxwell’s equations due to the linearity imposed on the constitutive relationships. Each component is then a monochromatic wave

\[
\mathbf{E} (r, t) = \tilde{a} (r) \exp [i (\phi (r) - \omega t)] .
\]
The Poynting vector in the amplitude and phase representation is then

\[
S = \frac{1}{\omega \mu} (\tilde{a} \cdot \tilde{a}^*) \nabla \phi + \frac{1}{2\omega \mu} \left[ - (\tilde{a} \cdot \nabla \phi) \tilde{a}^* + i \tilde{a} \times (\nabla \times \tilde{a}^*) + c.c. \right].
\]

The normalized intensity Poynting vector \( S_N = S / (\tilde{a} \cdot \tilde{a}^*) \) has the form of the gradient of a scalar function \( \phi \) plus a vector field which resembles the Helmholtz decomposition proposed by Paganin and Nugent [83]. In order to establish an exact equivalence, it would be necessary to identify the vector field part that is divergence free. It is worth pointing out that this derivation does not impose the scalar theory condition of a scalar wave field with its concomitant definition of flow [84], [85].

The Poynting theorem for a monochromatic vector wave of arbitrary form propagating in linear inhomogeneous media without charges nor absorption reads

\[
\nabla \cdot \left[ \frac{1}{\omega \mu} (\tilde{a} \cdot \tilde{a}^*) \nabla \phi + \frac{1}{2\omega \mu} \left[ - (\tilde{a} \cdot \nabla \phi) \tilde{a}^* + i \tilde{a} \times (\nabla \times \tilde{a}^*) + c.c. \right] \right] = 0.
\]

From this expression it follows that in order to derive the phase, up to a constant, it is mathematically necessary to know the complex vector amplitude \( \tilde{a}(r) \) spatial dependence. The complex nature of the amplitude in this context refers to the state of polarization of the wave [73]. Let us impose the restriction of a real amplitude \( a(r) \), which implies that only linearly polarized fields are allowed. The Poynting vector is then \( S = \frac{1}{\omega \mu} [(a \cdot a) \nabla \phi - (a \cdot \nabla \phi) a] \). The term \( a \times (\nabla \times \tilde{a}^*) \) is therefore only relevant in the presence of elliptically polarized light and is possibly connected with vortex structures.

In the absence of sources with \( \varepsilon \) real (non absorbing medium), the term \( a \cdot \nabla \phi = 0 \) and the flow is thus

\[
S = \frac{1}{\mu} (a \cdot a) \frac{\nabla \phi}{\omega}.
\]

This expression is the general form of Poynting’s vector in terms of amplitude and phase variables for a linearly polarized wave propagating in an inhomogeneous transparent medium. Poynting’s theorem for a linearly polarized monochromatic wave with arbitrary wavefront reads

\[
\nabla \cdot \left( \frac{1}{\mu \omega a \cdot a} \nabla \phi \right) = 0.
\]

This equation may be expanded to yield

\[
a \cdot a \nabla^2 \phi + \nabla \phi \cdot \nabla (a \cdot a) - a \cdot a \nabla \phi \cdot \nabla \ln \mu = 0.
\]

Poynting’s vector (113) as well as the conservation equation (112) have been derived without making any assumptions on the wave vector magnitude. However, these expressions have long been known in the short wavelength limit where the optical path function \( S \) satisfies the eikonal equation \( \nabla S \cdot \nabla S = \varepsilon \mu \). The eikonal result is be obtained from (105) if the amplitude second order space and time derivatives are neglected. The transport equation (115) is also obtained in geometrical optics from the second order wave equations
in the limit where only linear terms in the wavelength are considered (only L-terms are retained in Born and Wolf’s terminology). However, the eikonal derivation neglects the amplitude derivatives and the inhomogeneity of the medium on a wavelength scale. As it is well known, such an approximation describes refractive phenomena but does not model wave phenomena such as interference and diffraction nor propagation through an abrupt interface.

In the present derivation, these equations have been obtained without invoking the short wavelength limit. The price that we have paid is that the amplitudes must be real. Therefore, Poynting’s vector and the conservation equation in the form (114) and (115) are exact for linearly polarized wave fields but correspond to the short wavelength approximation for fields with arbitrary elliptical polarization. The corresponding equations exact for vector waves with arbitrary polarization are given by (111) and (112).

4.4.1 Irradiance propagation equation

The time independent conservation equation may be rewritten in the form of a propagation equation. To this end, let the gradient operator be separated into a transverse and a longitudinal operator \( \nabla = \nabla_T + \frac{\partial}{\partial z} \hat{e}_z \). The transverse part in Cartesian or cylindrical coordinates is correspondingly given by \( \nabla_T = \frac{\partial}{\partial x} \hat{e}_x + \frac{\partial}{\partial y} \hat{e}_y \) or \( \nabla_T = \frac{\partial}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \theta} \hat{e}_\theta \). The divergence equation (114) for a non magnetic medium then resembles the form of a continuity equation

\[
\nabla_T \cdot (I \nabla_T \phi) + \frac{\partial}{\partial z} \left( I \frac{\partial \phi}{\partial z} \right) = 0, \tag{116}
\]

where \( I = a \cdot a \) and \( z \) plays the role of \( t \) as is customary in the Hamiltonian analogy between mechanics and optics [73]. The wave vector magnitude squared is defined by

\[
\nabla \phi \cdot \nabla \phi = k^2 = \nabla_T \phi \cdot \nabla_T \phi + \left( \frac{\partial \phi}{\partial z} \right)^2.
\]

This definition in terms of the wave’s phase subordinates the relationship \( k \cdot k = \mu \varepsilon \omega^2 \) as an approximate expression valid in the eikonal limit. The derivative of the phase in the \( z \) direction is then \( \frac{\partial \phi}{\partial z} = \left( k^2 - \nabla_T \phi \cdot \nabla_T \phi \right)^{1/2} \). Let the wave propagation be preferentially in the \( z \) direction in order to impose the paraxial approximation

\[
\left( \frac{\partial \phi}{\partial z} \right)^2 \gg \nabla_T \phi \cdot \nabla_T \phi,
\]

so that the binomial expansion of the root yields \( \frac{\partial \phi}{\partial z} = k - \frac{1}{2k} \nabla_T \phi \cdot \nabla_T \phi + \ldots \). Retaining only the first term of the above expression in the continuity equation (116) yields

\[
\nabla_T \cdot (I \nabla_T \phi) + \frac{k I}{\partial z} + I \frac{\partial k}{\partial z} = 0. \tag{117}
\]

This equation represents the propagation of the irradiance in an inhomogeneous medium in the paraxial approximation. In vacuo, the wave vector magnitude is constant provided
that diffraction or interference effects may be neglected. Within the present approximation \( \frac{\partial k}{\partial z} = 0 \), which amounts to neglecting second order derivatives of the phase in the direction of propagation. The irradiance transport equation then reads

\[
I \nabla^2 \phi + \nabla T I \cdot \nabla T \phi + k \frac{\partial I}{\partial z} = 0.
\]

(118)

This particular result corresponds to that originally obtained by Teague and subsequent workers [86]. This expression has been used to retrieve the phase from intensity measurements in optical testing [80, p. 311] and other inverse source problems [79].

### 4.5 Nonlinear amplitude equation - vector wave equation

Consider the electric field wave equation that arises from Maxwell’s equations for a linear isotropic dispersion-less medium without charges in SI units,

\[
\nabla^2 E - \mu \varepsilon \frac{\partial^2 E}{\partial t^2} = -\nabla (E \cdot \nabla \ln \varepsilon) - \nabla \ln \mu \times (\nabla \times E).
\]

(119)

Allow for an inhomogeneous non absorbing material so that the permittivity and permeability are real space dependent quantities. Let the electric field be written in terms of the complex function (110) but with the restriction of a linear polarization that implies a real amplitude \( a(r) \). The real part of the wave equation yields

\[
\nabla^2 a - (\nabla \phi \cdot \nabla \phi) a + \mu \varepsilon \omega^2 a = \mathcal{F}_E,
\]

(120)

where the electric field source terms due to the inhomogeneity of the medium are

\[
\mathcal{F}_E = -\nabla [a \cdot (\nabla \ln \mu)] + (a \cdot \nabla) \nabla \ln \mu + (\nabla \ln \mu \cdot \nabla) a.
\]

In order to obtain an equation in terms of the intensity instead of the amplitude, evaluate the outer product of the amplitude \( a \) with (120). The equation in terms of intensity and phase is then

\[
\frac{1}{2} \nabla^2 I - \frac{1}{4} \sum_{m=1}^{3} I_m \nabla I_m \cdot \nabla I - I (\nabla \phi \cdot \nabla \phi) + \mu \varepsilon \omega^2 k_0^2 I = \mathcal{F}_E \cdot a,
\]

where the intensity and its projections are defined by \( I = a \cdot a \) and \( I_m = a_m^2 \); the wave vector magnitude in vacuo is \( k_0^2 = \mu_0 \varepsilon_0 \omega^2 \) and the relative permittivity and permeability are defined by \( \varepsilon = \varepsilon_0 \varepsilon_r \) and \( \mu = \mu_0 \mu_r \). In the case of a scalar field propagating in a homogeneous medium,

\[
\frac{1}{2} \nabla^2 I - \frac{1}{4} I \nabla I \cdot \nabla I - I (\nabla \phi \cdot \nabla \phi) + n^2 k_0^2 I = 0.
\]

If the amplitude (or intensity) of the wave is independent of the wave vector \( k_0 \), then the above equation must be satisfied by the first and second pair of terms separately. In this
case, the latter pair leads to the eikonal equation. Therefore, in a homogeneous medium, any linearly polarized time-harmonic wave whose amplitude is wave vector independent obeys rigorously the laws of geometrical optics. The ample validity of the eikonal approximation to waves where the vector potential is associated with a scalar complex potential has been discussed in detail by Green and Wolf [72].

On the other hand, the imaginary part of the wave equation (119) in amplitude and phase variables is

\[ a \nabla^2 \phi + 2 (\nabla \phi \cdot \nabla) a = - (a \cdot \nabla \ln \epsilon) \nabla \phi + \nabla \ln \mu \times (a \times \nabla \phi). \]

Imposing the condition of a transparent medium, the product \( a \cdot \nabla \phi \) stemming from the triple vector product expansion is zero. The imaginary part of the wave equation becomes

\[ \nabla \cdot \left( \frac{1}{\mu} a \cdot a \nabla \phi \right) = 0. \]

This equation is equal to Eq. (114) except for a factor of \( 1/\omega \). Nonetheless, since \( \omega \) is constant for a monochromatic wave, these two results are equivalent. The conservation equation for a coherent field may thus be derived using either Poynting’s theorem or the wave equation as the starting point. However, if the wave amplitude and frequency are time dependent, then these two procedures do not necessarily lead to the same result. If the paraxial wave equation [87, p.626] \( \nabla_T a + 2ik(\partial/\partial z) a = 0 \) with \( E(\mathbf{r}) = a(\mathbf{r}) \exp(ikz) \), is used as the starting point of the derivation rather than the wave equation (119), then the procedure outlined in this section leads to the irradiance transport equation (118).

In order to illustrate the decoupling of the two differential equations obtained for the amplitude and phase, consider the one dimensional propagation of the field (say in the \( z \) direction) in a non magnetic medium. The conservation equation leads to an invariant

\[ Q = a \cdot a \frac{\partial \phi}{\partial z}. \]  

(121)

This is the vector version of expression ((95)). For a non magnetic medium, the wave equation for a linearly polarized monochromatic field in terms of the amplitude and phase variables is then

\[ \frac{\partial^2 a}{\partial z^2} - a \left( \frac{\partial \phi}{\partial z} \right)^2 + \mu_0 \omega^2 \epsilon(a) a = 0. \]

From these two equations, an equation for the phase or the amplitude may be readily obtained. The nonlinear vector amplitude equation is

\[ \frac{\partial^2 a}{\partial z^2} - \frac{a Q}{(a \cdot a)^2} + \mu_0 \omega^2 \epsilon(a) a = 0, \]  

(122)

which, for a scalar amplitude acquires the form of the nonlinear amplitude equation (6).
4.6 Ultrashort pulses

The continuity equation associated with energy conservation in classical scalar waves or a positive definite probability density in second order differential equations in quantum mechanics may be derived in a variety of ways. One such procedure is to multiply the dispersion-less wave equation by the temporal derivative of the wave function [88, p.71]. Upon rearrangement of terms, the conservation equation reads
\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \frac{1}{v^2} \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{1}{2} \nabla \psi \cdot \nabla \psi \right] + \nabla \cdot \left( -\frac{\partial \psi}{\partial t} \nabla \psi \right) = 0.
\]

Thus a density and a flux are defined as
\[
\mathcal{E}_d = \frac{1}{2} \frac{1}{v^2} \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{1}{2} \nabla \psi \cdot \nabla \psi, \quad \mathbf{S}_d = -\frac{\partial \psi}{\partial t} \nabla \psi. \quad (123)
\]

These variables are usually translated into quantities with energy and energy flow units through multiplication by the adequate quantity. It is customary to associate the two terms in the energy density as the sum of kinetic and potential energy [89, pp.161-162].

The purpose in this section is to compare these expressions with their counterparts obtained from the complementary fields approach. The first issue that has already been pointed out is that whereas the density \( \mathcal{E}_d \) is positive definite, the densities that arise from the complementary fields are not positive definite but under restricted circumstances. Nonetheless, it should be mentioned that the density \( \mathcal{E}_d \) also presents certain drawbacks, for example under Lorentz transformations where it does not lead to a relativistically invariant definition of the integrated probability [90, pp.89-90].

The complementary field evaluated with the derivative field yields a density and flow that has the same spatial and temporal parities as the energy density and flow. Namely \( \mathcal{E}_d \) and \( \psi^\rho (\text{DF}) \) are even under time and space inversion whereas \( \mathbf{S}_d \) and \( \psi^\rho (\text{DF}) \) are odd in either case. For this reason, we shall compare these quantities hereafter. The energy density in terms of the amplitude and phase variables (99) is given by
\[
\mathcal{E}_d = \frac{1}{2} a^2 \left( \frac{1}{v^2} \left( \frac{\partial s}{\partial t} \right)^2 + \nabla s \cdot \nabla s \right) \sin^2 s + \frac{1}{2} \frac{1}{v^2} \left( \frac{\partial a}{\partial t} \right)^2 + \nabla a \cdot \nabla a \right) \cos^2 s \quad (124)
\]

The corresponding flow is
\[
\mathbf{S}_d = \left( -a^2 \nabla \frac{\partial s}{\partial t} \sin^2 s - \nabla a \frac{\partial a}{\partial t} \cos^2 s + \frac{a}{2} \left( \nabla s \frac{\partial a}{\partial t} + \nabla a \frac{\partial s}{\partial t} \right) \sin (2s) \right). \quad (125)
\]

The density \( \psi^\rho (\text{DF}) \) and flow \( \psi^\rho (\text{DF}) \) given by Eqs. (108) and (109) respectively have a somewhat similar structure. Nonetheless, the (DF) quantities involve terms that do not exhibit a phase dependent oscillation as well as second order derivatives. In the particular
case of a plane wave, since the amplitude is constant and the phase is linear in the time and space variables, the energy density and flow are then

$$\mathcal{E}_d = \frac{1}{2} a_0^2 \left( \frac{1}{v^2} \omega_0^2 + \mathbf{k} \cdot \mathbf{k} \right) \sin^2 (\mathbf{k} \cdot \mathbf{r} - \omega t), \quad \mathbf{S}_d = a_0^2 \omega_0 \sin^2 (\mathbf{k} \cdot \mathbf{r} - \omega t) \mathbf{k};$$

so that even in this particular condition they are time dependent. In contrast, the complementary field quantities evaluated with the derivative field yield time independent results

$$\psi_\rho^{(DF)} = a_0^2 \omega_0^2, \quad \nabla \psi_\rho^{(DF)} = a_0^2 \omega_0 \mathbf{k},$$

just as in the orthogonal trajectories scheme (104). The underlying reason for this state of affairs is that the kinetic and potential energy terms in (123) correspond to the sum of in phase fields whereas the terms in the complementary fields density (107) correspond to the addition of out of phase fields.

### 4.6.1 Averages

Rather than the instantaneous time dependent energy density and flow what is frequently measured is the average of these quantities. If the change of the amplitude and the phase temporal derivative are negligible over a period, averages may be performed over the fast varying trigonometric functions. Nonetheless, it should be stressed that this approximation is no longer be valid for ultra short pulses as those presently attainable in the femtosecond regime in the optical region [91]. A constant time derivative of the phase within a period is approximated by

$$s(r, t) = \phi(r) - \omega t, \quad \text{where } \omega \text{ is constant over a period.}$$

The average per unit time of the squared trigonometric functions is thus

$$\omega/2\pi \int_0^{2\pi/\omega} \sin^2 (\phi(r) - \omega t) \, dt = 1/2. $$

The average energy density from (124) is then

$$\langle \mathcal{E}_d \rangle = \frac{1}{4} \left( \frac{1}{v^2} a_0^2 \frac{\partial s}{\partial t} \right)^2 + \frac{1}{v^2} \left( \frac{\partial a}{\partial t} \right)^2 + a^2 \nabla s \cdot \nabla s + \nabla a \cdot \nabla a \right), $$

whereas the average flow is

$$\langle \mathbf{S}_d \rangle = -\frac{1}{2} \left( a^2 \nabla s \frac{\partial s}{\partial t} + \nabla a \frac{\partial a}{\partial t} \right).$$

These expressions are often used in a scalar representation of electromagnetic fields [72]. The first and second pair of terms in (126) are associated with electric and magnetic energy densities respectively. Radiometry in the Walther, Marchand and Wolf (WMW) formulation [84, pp.278-293] also use these definitions of density and flow that in terms of a complex scalar \( \tilde{\psi} = a \exp (is) \) are given by [85, p.7]

$$\langle \mathcal{E}_d \rangle = \frac{1}{2} \nabla \tilde{\psi} \cdot \nabla \tilde{\psi}^* + \frac{1}{2} v^2 \frac{\partial \tilde{\psi}}{\partial t} \frac{\partial \tilde{\psi}^*}{\partial t}, \quad \langle \mathbf{S}_d \rangle = -\frac{1}{2} \left( \frac{\partial \tilde{\psi}}{\partial t} \nabla \tilde{\psi}^* + \frac{\partial \tilde{\psi}^*}{\partial t} \nabla \tilde{\psi} \right).$$

(128)
These results should be contrasted with the complementary fields complex representation (97) where no averaging is being performed. On the other hand, the average of the complementary fields density using the derivative field for the linearly independent solution (108) reduces to

$$\langle \psi_p^{(DF)} \rangle = a^2 \left[ \left( \frac{\partial s}{\partial t} \right)^2 - \frac{1}{2} \frac{\partial^2 \ln a}{\partial t^2} \right] = a^2 \left( \frac{\partial s}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial a}{\partial t} \right)^2 - \frac{1}{2} \frac{a^2}{2} \frac{\partial^2 a}{\partial t^2}$$

and the average flow from (109) is

$$\langle \triangleright \psi_p^{(DF)} \rangle = -a^2 \nabla s \frac{\partial s}{\partial t} - \frac{1}{2} \nabla a \frac{\partial a}{\partial t} + \frac{1}{2} a \nabla \left( \frac{\partial a}{\partial t} \right)$$

The WMW approach employs the spectral flow vector as a starting point for calculating all radiometric quantities. The spectral flow vector is defined as the flow vector (128) restricted to a monochromatic field. Since the monochromatic condition requests a time independent amplitude, the spectral flow vector in the WMW (127) and the complementary fields (130) formalisms become identical but for a factor of 1/2 that should be included in the latter definition. However, the average energy density (126) is not equal to the average of the complementary fields density (129) in the monochromatic case. Nonetheless, both quantities only differ by a time independent function that preserves the same conservation equation. If the field is further restricted to a plane wave, then the average density in the two formalisms is the same provided that the complementary fields density is scaled by a factor of \((2v^2)^{-1}\). This factor has not been included in the complementary field definitions in order to allow for the possibility of dealing with appropriate designations for a homogeneous medium with dispersion or an inhomogeneous medium without dispersion (See Eq. (87) and thereafter).

The situation becomes quite different in the presence of finite wave-trains. The amplitude time derivatives are increasingly important as the wave is shortened in time. Furthermore, for very short pulses a coupling of spatial and temporal effects becomes important even for propagation in a non dispersive medium [92, pp.39-40]. Important differences are thus expected in these two schemes in the presence of very short pulses even for the averaged quantities.

## 5 Conclusions

The nonlinear amplitude equation has been studied in three different Ermakov systems. A simple invariant derivation for the classical TDHO has been presented. It has been shown that a detailed understanding of the nonlinear superposition principle may be invoked to find novel solutions to the nonlinear amplitude equation (21). This procedure is suitable for the development of analytical solutions with more general parameters. The indeterminacy in amplitude and phase variables in non-propagating phenomena has been discussed. The relationship between adiabatic and exact invariants has been addressed.
The QM-TDHO has been solved using the invariant formalism together with unitary transformations. The amplitude and phase operators for the time dependent case have been presented. The quantum invariant has been expressed in terms of these operators (61). Minimum uncertainty states in the time dependent case have been considered. The frequency step function has been analytically solved.

Electromagnetic wave propagation has been studied in deterministic inhomogeneous media. Numerical solutions have been presented in the sub-wavelength regime with 1+1 dimensions. The one dimensional invariant formalism has been extended to 3+1 dimensions. The complementary field has been evaluated using two different criteria. The relationship between amplitude and phase variables allows for the phase retrieval from intensity measurements. The complementary fields formalism has been shown to lead to radiometric quantities (108, 129) and (109, 130) that differ for ultra-short pulses from the usual optical radiometric theory.

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